7 *Rate-of-Return Analysis*

After this chapter you will be able to:

1. state what the internal rate of return i^* is, and read it off a present-worth profile as the rate where $PW(i^*) = 0$
2. compute i^* by direct solution, and by trial and error with one linear interpolation
3. classify an investment as simple or nonsimple by counting the sign changes in its cash flow
4. decide whether to accept a simple investment by comparing i^* with the MARR
5. recognize a mixed investment with the net-investment test, and compute its external rate of return (MIRR)
6. explain why a rate cannot rank mutually exclusive alternatives, and choose among them by the incremental method

Present-worth analysis (chapter 5) answers an investment question in money at time zero. Annual-equivalence analysis (chapter 6) answers the same question in money per year. Managers, however, often ask for the answer as a percentage: *what return does this project earn?* That percentage is the rate of return. It is the most widely quoted figure in investment decisions, and it is also the easiest to misuse. This chapter defines the internal rate of return, develops three ways to compute it, classifies investments so that the computed rate can be trusted, and states the decision rule that compares the rate against the MARR. It then turns to the comparison of alternatives, where the rate of return fails if it is used directly, and develops the incremental method that repairs it.

Rates, hurdles, and the bigger project.

Q1. You invest 5,000 SAR today and receive 12,000 SAR at the end of year 6. What rate of return does this investment earn?

Q2. Does a project's internal rate of return depend on the firm's MARR?

Q3. Project A costs 10,000 SAR and earns 25%. Project B costs 16,000 SAR and earns 19%. The rate of return on the incremental investment B−A is 14%, and the MARR is 12%. You must choose one. Which one?

Answers.

Q1. Set $5{,}000(1+i)^6 = 12{,}000$, so $(1+i)^6 = 2.4$ and $i = 2.4^{1/6} - 1 = 15.71\%$. One amount in, one amount out: the rate is found by taking a root, exactly as in Chapter 2.

Q2. *No.* The rate is computed from the project's own cash flows only; this is why it is called *internal*. The MARR enters only at the decision step, when the computed rate is compared against it.

Q3. *B.* The extra 6,000 SAR invested in B earns 14%, which is above the 12% hurdle, so the larger investment is justified. Choosing A because its own rate is higher is the classic error that this chapter corrects.

Lecture 17. This section is covered by Lecture 17.

Reference. Park, 3rd ed., Chapter 7, Sections 7.1–7.3 (pp. 296–313).

Rate of return. The interest rate earned on the unrecovered balance of an investment, such that the final cash flow brings the balance exactly to zero.

Internal rate of return (i^*). The rate that makes the present worth of a project's cash flows equal to zero: $PW(i^*) = 0$. ''Internal'' because it depends only on the project's own cash flows.

7.1 *What the Rate of Return Is*

The *internal rate of return*, written i^*, is the interest rate that makes a project break even:

$$PW(i^*) = \sum_{n=0}^{N} A_n (1+i^*)^{-n} = 0, \tag{7.1}$$

where A_n is the net cash flow at the end of year n, negative for money out and positive for money in, and N is the last year of the project. Each flow is discounted back to year zero, and i^* is the rate at which the discounted flows sum to zero.

Insight. The rate is best read through the loan table of Chapter 2. In a loan, the balance grows by interest each period and drops with each payment. A project is the same bookkeeping in reverse: the firm puts capital in, and the project pays interest on the *unrecovered balance* at the rate i^*, until the final cash flow closes the account at exactly zero. So i^* is the interest rate the project itself generates on the money still tied up in it. It depends only on the project's cash flows, not on any outside rate, which is why it is called internal.

Two consequences follow directly from the definition. First, at $i = i^*$ the net present worth is zero and the net future worth is zero: the project repays its capital and earns exactly i^*, with no surplus. Second, you have already computed a rate of return. Whenever one amount goes in, one amount comes out, and the connecting rate is solved for, that rate is an internal rate of return.

In practice. Venture investors quote IRR for the same reason managers do. A fund that reports only a money multiple, say 3x, hides how long the capital was held; 3x in three years and 3x in twelve years are very different outcomes, and the IRR is the scale that separates them.

7.1.1 *The present-worth profile*

Picture the present worth as a function of the interest rate. For an investment, money out first and returns later, the curve *falls* as the rate rises. The reason is discounting: a higher rate shrinks the far-away returns much more than it affects the up-front cost, which sits at time zero and is not discounted at all. Somewhere the falling curve crosses zero, and that crossing point is i^*. Figure 7.1 shows the shape.

MARR. The minimum attractive rate of return: the lowest return the firm accepts, because its capital can earn that much elsewhere. The hurdle used in every decision rule of this book.

PW profile. The plot of $PW(i)$ against the interest rate i. For an investment, the curve falls as i rises and crosses zero at i^*.

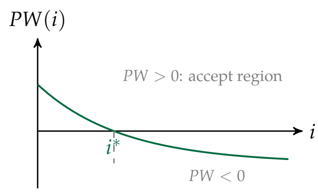

Figure 7.1. The PW profile of an investment. Left of i^* the present worth is positive; right of it, negative.

The profile also previews the decision rule. To the left of i^* the present worth is positive, so any MARR below i^* means accept. To the right, the present worth is negative, so any MARR above i^* means reject. Sketching this curve is a good exam habit: even a rough sketch with two computed points on it shows where the root sits. Please note that the single-crossing shape is guaranteed only for a *simple* investment, a term defined in Section 7.4.

7.2 *Finding i^* by Direct Solution*

When the cash flow is short, Equation (7.1) can be solved by algebra. Two cases cover most of them: a single future flow, and a project with two future periods.

Example 7.1 — Single future flow: take a root

A drill-bit patent license costs 8,000 SAR today and pays a single 20,000 SAR royalty at the end of year 5. Find i^*.

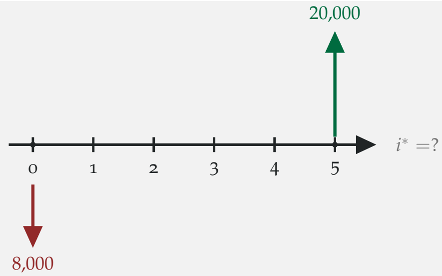

Figure 7.2. One amount out at year 0, one amount in at year 5; the rate connecting them is i^*.

Solution.

Set the present worth to zero:

$$PW(i^*) = -8{,}000 + 20{,}000\,(1+i^*)^{-5} = 0 \quad \Longrightarrow \quad (1+i^*)^5 = \frac{20{,}000}{8{,}000} = 2.5.$$

The unknown rate is the base, so take the fifth root:

$$i^* = 2.5^{1/5} - 1 = 1.2011 - 1 = \boxed{20.11\%}.$$

This is the same procedure as in Chapter 2: two amounts and a time span give the rate by taking a root. No factor table is involved.

Example 7.2 — Two future periods: a quadratic in $x = (1+i)^{-1}$

A used screening deck is bought for 10,000 SAR. It nets 6,000 SAR in year 1 and 7,200 SAR in year 2, and is then scrapped at zero value. Find i^*.

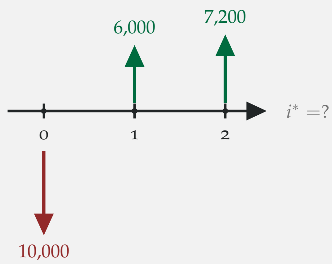

Figure 7.3. A two-period project; its $PW = 0$ equation is a quadratic.

Solution.

Substitute $x = (1+i)^{-1}$, the one-year discount factor. The year-1 flow is multiplied by x once and the year-2 flow by x^2, so the present-worth equation becomes an ordinary quadratic in x:

$$-10{,}000 + 6{,}000\,x + 7{,}200\,x^2 = 0.$$

Apply the quadratic formula and keep the positive root, because a discount factor must be positive. The discriminant is $6{,}000^2 + 4\,(7{,}200)(10{,}000) = 324{,}000{,}000$, whose square root is 18,000:

$$x = \frac{-6{,}000 + 18{,}000}{2\,(7{,}200)} = \frac{12{,}000}{14{,}400} = 0.8333.$$

Convert back:

$$1 + i^* = \frac{1}{0.8333} = 1.20 \quad\Longrightarrow\quad \boxed{i^* = 20.00\%}.$$

Please practice this substitution until it is automatic. Two-period rate problems appear often, and the quadratic route is the fastest exact method.

7.3 *Finding i* by Trial and Error*

Most projects have too many periods for a formula, so the general method is trial and error with the factor tables. Evaluate $PW(i)$ at trial rates until the sign changes, then interpolate once along the straight line between the two bracketing points:

$$i^* \;\approx\; i_L + (i_H - i_L)\,\frac{PW(i_L)}{PW(i_L) - PW(i_H)}, \tag{7.2}$$

where i_L is the lower trial rate with $PW(i_L) > 0$ and i_H is the higher trial rate with $PW(i_H) < 0$. Read the formula as: start at the lower bracket, then add the bracket width times the fraction of the total present-worth swing that sits above zero.

Example 7.3— Trial and error with linear interpolation

A conveyor drive replacement costs 75,000 SAR now and saves 24,400 SAR per year for 5 years. Find i^*.

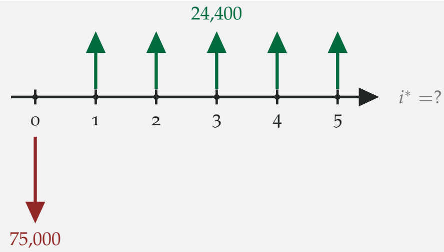

Figure 7.4. An investment now, recovered by five equal annual savings.

Solution.

Step 1: write the present worth in factor notation.

$$PW(i) = -75{,}000 + 24{,}400\,(P/A,\, i,\, 5).$$

Step 2: bracket a sign change.

Try rates from the tables, one at a time (Table 7.1).

try i	15%	18%	20%
$(P/A, i, 5)$	3.3522	3.1272	2.9906
$PW(i)$, SAR	+6,793.68	+1,303.68	−2,029.36

Table 7.1. Trial rates for Example 7.3. The sign of PW changes between 18% and 20%.

At 15% the present worth is positive, so the true rate is higher. At 18% it is still positive; at 20% it turns negative. The rate is bracketed between 18% and 20%.

Step 3: interpolate once.

Apply Equation (7.2) with $i_L = 18\%$ and $i_H = 20\%$:

$$i^* \approx 18\% + 2\% \times \frac{1{,}303.68}{1{,}303.68 - (-2{,}029.36)} = 18\% + 2\%\,(0.391) = \boxed{18.78\%}.$$

The exact root, found numerically, is 18.76%; the interpolated answer differs only in the second decimal. Figure 7.5 plots the profile with the three trial points on it.

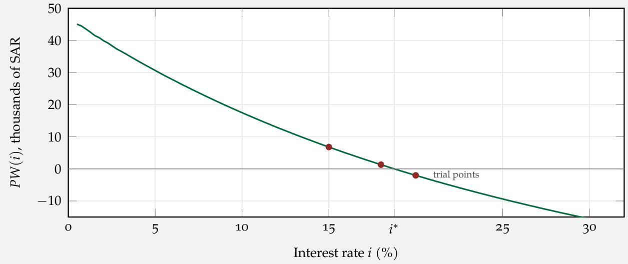

Figure 7.5. The PW profile of the conveyor drive. The curve falls with i and crosses zero once, at $i^* = 18.76\%$, between the bracketing trial points.

The routine is always the same: bracket a sign change with the tables, then one straight-line interpolation. Two brackets and one interpolation is all that is expected on graded work.

Insight. The interpolation is an approximation, because the true profile bends slightly. For a typical investment the profile is convex, so the straight chord lies below the curve and the interpolated rate comes out slightly *above* the true root (18.78% against 18.76% in Example 7.3). Keeping the brackets a few percent apart keeps this error in the second decimal.

A spreadsheet computes the same rate directly. `=IRR(range)` takes the full cash flow series, including the year-0 flow, and returns i^*; `=RATE(N, A, P)` handles the special case of a uniform series against a single present amount. Use the spreadsheet to check hand work, not to replace it: the exam expects the bracket-and-interpolate routine.

7.4 *Simple and Nonsimple Investments*

Before trusting any computed rate, classify the investment. Scan the net cash flow signs from left to right and count the changes.

- **One sign change**: a *simple* investment. It has exactly one rate that zeroes the present worth, so the number you compute is the internal rate of return.
- **More than one sign change**: a *nonsimple* investment. It *may* have more than

Simple investment. A cash flow series with exactly one sign change. It has a unique i^*, and that i^* is the internal rate of return.

Nonsimple investment. A series with more than one sign change. It may have more than one value of i^*, and the rate can lose its meaning.

one root. May, not must: some nonsimple projects still have a single rate. The point is that uniqueness can no longer be assumed.

- **No sign change**: no rate of return exists. There is no invested balance to earn a return on.

Insight. The count comes from algebra. With $x = (1+i)^{-1}$, the equation $PW = 0$ is a polynomial in x whose coefficients are the cash flows. Descartes' rule of signs says the number of positive real roots is at most the number of sign changes in the coefficients. One sign change therefore allows one root; several sign changes allow several. The classification is a five-second scan that tells you how many answers the equation is allowed to have.

Example 7.4 — Classifying three cash flow series

Classify each project in Table 7.2 as simple, nonsimple, or having no rate of return.

	$n = 0$	1	2	3	sign changes
A	$-1{,}000$	$+500$	$+800$	$+900$	1
B	$-1{,}000$	$+3{,}900$	$-5{,}030$	$+2{,}145$	3
C	$+1{,}200$	$+500$	$+300$	$+100$	0

Table 7.2. Sign-change classification for Example 7.4 (amounts in SAR).

Solution.

Project A starts negative and turns positive once: one sign change, so a *simple* investment with a unique i^*, and that i^* is the IRR.

Project B runs $-, +, -, +$: three sign changes, so *nonsimple*. Its diagram makes the alternation visible:

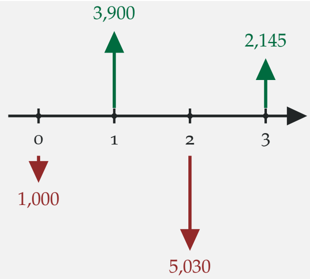

Figure 7.6. Project B alternates sign three times; up to three roots are possible.

In fact this series has three roots: $PW(i) = 0$ at $i = 10\%$, 30%, and 50%. None of the three is "the" return of the project.

Project C has no negative flow at all: zero sign changes, so $\boxed{\text{no rate of return exists}}$. There is no investment to recover, and $PW(i) > 0$ at every rate, so the equation has no root. Take the money, but do not quote a percentage on it.

In practice. Sign patterns are engineering facts before they are financial ones. A mid-life rebuild, a scheduled relining, or a battery replacement plants a negative flow in the middle of the series, and a reclamation or well-plugging obligation plants one at the end, so the maintenance and closure plan fixes the pattern before any rate is computed.

Please make this scan a habit. It takes five seconds, and it decides whether the decision rule of the next section applies.

7.5 The IRR Decision Rule

For a *simple* investment with its unique i^*, the decision rule is:

- $i^* >$ MARR: **accept**;
- $i^* =$ MARR: indifferent;
- $i^* <$ MARR: **reject**.

This is not a new criterion in disguise. Look back at the falling profile in Figure 7.1: a MARR below i^* is exactly the region where $PW(\text{MARR}) > 0$. So for simple investments the rate rule and the present-worth rule of chapter 5 always agree:

$$\text{MARR} < i^* \quad \Longleftrightarrow \quad PW(\text{MARR}) > 0. \tag{7.3}$$

What the rate adds is a different reading of the same fact. A project that earns 18.78% should be taken by a firm whose capital earns 12% elsewhere, and declined by a firm whose alternatives earn 20%. Same project, different hurdle, different verdict, and both verdicts are correct. The rate is a property of the project; the decision also depends on the firm.

Example 7.5 — Applying the decision rule

A dewatering pump upgrade costs 50,000 SAR and saves 13,000 SAR per year in energy for 6 years, with no salvage value. The MARR is 12%. Classify the investment, find i^*, and state the decision.

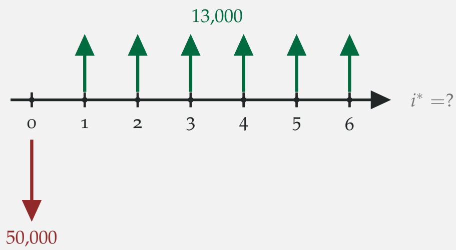

Figure 7.7. The pump upgrade: one outflow, six equal savings. One sign change, so a simple investment.

Solution.

Step 1: classify.

The signs run $-, +, +, +, +, +, +$: one change, so the investment is simple and its i^* is unique.

Step 2: bracket and interpolate.

Write $PW(i) = -50{,}000 + 13{,}000\,(P/A,\, i,\, 6)$ and try two rates:

$$PW(14\%) = -50{,}000 + 13{,}000\,(3.8887) = +553.10 \text{ SAR},$$
$$PW(15\%) = -50{,}000 + 13{,}000\,(3.7845) = -801.50 \text{ SAR}.$$

The sign change brackets the rate between 14% and 15%. Interpolate with Equation (7.2):

$$i^* \approx 14\% + 1\% \times \frac{553.10}{553.10 + 801.50} = 14\% + 1\%\,(0.408) = \boxed{14.41\%}.$$

(The exact root is 14.40%.)

Step 3: decide.

$i^* = 14.41\% > \text{MARR} = 12\%$, so $\boxed{\text{accept}}$. As a check, the present worth at the MARR is $-50{,}000 + 13{,}000\,(4.1114) = +3{,}448.20$ SAR, positive, the same verdict, as Equation (7.3) requires.

7.6 *Multiple Rates of Return*

The classification matters because a nonsimple investment can produce several rates at once, and then none of them means anything.

Example 7.6 — A strip mine with two rates of return

A strip-mining project costs 1,000,000 SAR now, returns 2,300,000 SAR in year 1, and requires a reclamation payment of 1,320,000 SAR in year 2. The MARR is 15%. Find every i^* and make the correct decision.

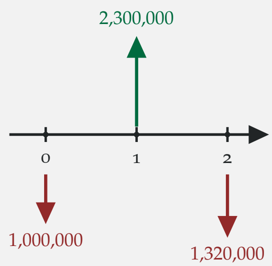

Figure 7.8. The strip mine: signs run $-, +, -$, two sign changes, so up to two roots.

Solution.

Step 1: classify.

The signs run $-, +, -$: two changes, so the project is nonsimple and up to two roots are possible.

Step 2: solve the quadratic.

Work in thousands of SAR and let $x = (1+i)^{-1}$:

$$-1{,}000 + 2{,}300\,x - 1{,}320\,x^2 = 0.$$

The discriminant is $2{,}300^2 - 4\,(1{,}320)(1{,}000) = 10{,}000$, whose square root is 100, so

$$x = \frac{2{,}300 \pm 100}{2\,(1{,}320)} = \frac{2{,}400}{2{,}640} \text{ or } \frac{2{,}200}{2{,}640} = \frac{10}{11} \text{ or } \frac{10}{12},$$

which gives

$$\boxed{i^* = 10\% \text{ and } 20\%}.$$

Both rates make the present worth exactly zero (Figure 7.9). Which one is "the" return? Neither. When more than one root exists, the unrecovered balance changes sign partway through the project, and the arithmetic behind the earned-rate interpretation breaks down.

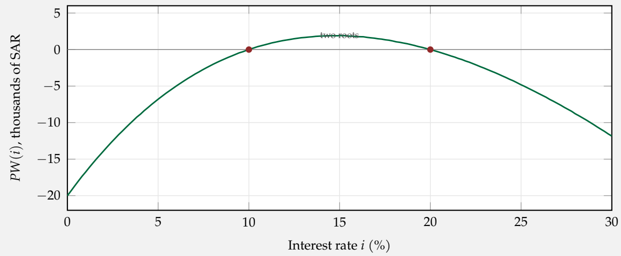

Figure 7.9. The strip mine's PW profile crosses zero twice, at 10% and 20%. Between the roots the present worth is barely positive (its peak is about +1,934 SAR at $i \approx 14.8\%$); outside them it is negative.

Step 3: decide by present worth at the MARR.

The rule for nonsimple investments is short: abandon the rate method. Decide by present worth (or annual equivalence, chapter 6) at the MARR, which never fails:

$$PW(15\%) = -1{,}000{,}000 + \frac{2{,}300{,}000}{1.15} - \frac{1{,}320{,}000}{(1.15)^2} = \boxed{+1{,}890 \text{ SAR}} > 0 \;\Rightarrow\; \text{accept.}$$

The margin is thin, but it is positive. The two roots were a warning sign, not an answer.

Please pay attention here, because this is a common trap. A computed rate from a nonsimple series looks exactly like a computed rate from a simple one. Only the sign-change scan tells you whether the number can be trusted. The next section adds two refinements for the strongest readers: a test that tells you when a computed rate still deserves trust, and a repaired rate for when it does not.

7.7 *When IRR Misleads: the Net-Investment Test and the External Rate of Return*

Reference. Park, 3rd ed., Chapter 7A (resolution of multiple rates of return).

Section 7.6 ended with blunt advice: when the series is nonsimple, abandon the rate and decide by present worth. That advice is always safe, but it leaves a question hanging. Managers still ask for a percentage, and some nonsimple projects still have a perfectly meaningful one. This section supplies the missing pieces: a five-minute test that sorts nonsimple projects into those whose i^* can be trusted and those whose cannot, and a repaired rate for the second kind.

Net-investment test. Track the project balance at i^*: $PB_0 = A_0$ and $PB_n = PB_{n-1}(1+i^*) + A_n$. If the balance stays at or below zero until the final year, the investment is *pure* and i^* keeps its meaning; if the balance turns positive along the way, the investment is *mixed* and i^* loses it.

7.7.1 *The net-investment test*

Recall the loan-table reading of Section 7.1: at the rate i^*, the project pays interest on the firm's unrecovered balance until the final flow closes the account at zero. Write that balance out year by year, at the rate being tested:

$$PB_0 = A_0, \qquad PB_n = PB_{n-1}\,(1+i^*) + A_n, \tag{7.4}$$

where A_n is the net cash flow of year n. A *negative* balance means the firm still has money invested in the project. A *positive* balance means the opposite: the project has handed back more than it owed, and the firm is temporarily holding the project's money. By the definition of i^*, the balance always ends at exactly $PB_N = 0$.

The test reads the signs of the intermediate balances:

- $PB_n \leq 0$ for every $n < N$: a **pure** investment. The firm's money stays inside the project for the whole life, earning i^* on the unrecovered balance, so the earned-rate reading survives — even if the cash flow series is nonsimple.
- $PB_n > 0$ for some $n < N$: a **mixed** investment. During the positive-balance years the money sits with the *firm*, and it earns whatever the firm's other opportunities pay, not i^*. The single-rate story breaks, and this is precisely the situation that breeds multiple roots.

Example 7.7 — Net-investment test on the strip mine

Apply the net-investment test to the strip mine of Example 7.6, whose flows are $-1{,}000{,}000$, $+2{,}300{,}000$, and $-1{,}320{,}000$ SAR and whose roots are $i^* =$

10% and 20%.

Solution.

Chain the balances of Equation (7.4) at each root. At $i^* = 10\%$:

$$PB_0 = -1{,}000{,}000,$$
$$PB_1 = -1{,}000{,}000\,(1.10) + 2{,}300{,}000 = +1{,}200{,}000,$$
$$PB_2 = +1{,}200{,}000\,(1.10) - 1{,}320{,}000 = 0.$$

At $i^* = 20\%$:

$$PB_1 = -1{,}000{,}000\,(1.20) + 2{,}300{,}000 = +1{,}100{,}000,$$
$$PB_2 = +1{,}100{,}000\,(1.20) - 1{,}320{,}000 = 0.$$

Both chains close at zero, as any root must, but in both the year-1 balance is *positive*: after the big year-1 receipt the project has overpaid the firm, and the firm holds the project's money for a year. The strip mine is a $\boxed{\text{mixed investment}}$, so neither 10% nor 20% is a rate the project ''earns'' — confirming the verdict of Example 7.6 by a different route. Note also what the balances reveal: the whole dispute is about what that +1,200,000 SAR earns during year 2, and the answer depends on the firm, not on the mine.

External rate of return (MIRR). Discount every negative flow to time 0 at the finance rate; compound every positive flow to year N at the reinvestment rate (the MARR); the single rate connecting the two lump sums is the MIRR. Accept when MIRR $\geq$ MARR.

7.7.2 *The external rate of return (MIRR)*

For a mixed investment, the repair is to stop pretending that the money the project hands back is reinvested at the unknown rate i^*, and to say explicitly what it earns instead. Two external rates do that. Money the firm receives early is reinvested at the firm's *reinvestment rate*, for which the natural choice is the MARR, since that is by definition what the firm's capital earns elsewhere. Money the firm must set aside for future outflows is priced at its *finance rate*, the rate at which the firm borrows or provisions; in this book, unless a problem states otherwise, both external rates are set equal to the MARR. The recipe:

1. Discount every negative flow back to time zero at the finance rate; call the magnitude of the total PV_-.

2. Compound every positive flow forward to year N at the reinvestment rate; call

the total FV_+.

3. One amount in, one amount out, N years apart — exactly the Chapter 2 situation, so take a root:

$$\text{MIRR} = \left(\frac{FV_+}{PV_-}\right)^{1/N} - 1. \tag{7.5}$$

The decision rule is the familiar one: accept when MIRR $\geq$ MARR. Because the construction collapses the series to a single outflow and a single inflow, the MIRR always exists and is always unique, no matter how many sign changes the original series had. And when both external rates equal the MARR, the MIRR verdict agrees with the present-worth verdict exactly, so nothing is lost by quoting it.

Example 7.8 — The strip mine, repaired with an external rate

Compute the MIRR of the strip mine ($-1{,}000{,}000$; $+2{,}300{,}000$; $-1{,}320{,}000$ SAR) with both external rates equal to the MARR of 15%, and decide.

Solution.

Step 1: park the negatives at time zero.

The year-0 outflow is already there; the year-2 reclamation is discounted at 15%:

$$PV_- = 1{,}000{,}000 + \frac{1{,}320{,}000}{(1.15)^2} = 1{,}000{,}000 + 998{,}110 = 1{,}998{,}110 \text{ SAR}.$$

Step 2: carry the positives to year 2.

The year-1 receipt is reinvested for one year at 15%:

$$FV_+ = 2{,}300{,}000\,(1.15) = 2{,}645{,}000 \text{ SAR}.$$

Step 3: take the root.

With $N = 2$,

$$\text{MIRR} = \left(\frac{2{,}645{,}000}{1{,}998{,}110}\right)^{1/2} - 1 = (1.3238)^{1/2} - 1 = \boxed{15.05\%}.$$

Since 15.05% $>$ 15% $=$ MARR, $\boxed{\text{accept}}$ — the same verdict as the present worth of $+$1,890 SAR in Example 7.6, and the same razor-thin margin: five hundredths of a point above the hurdle in rate language, a fifth of a percent of the investment in money language. The two roots of 10% and 20% said nothing useful; the MIRR says exactly what the present worth says, as a percentage.

Insight. Why the repair works. The multiple-root pathology comes from a hidden assumption inside Equation (7.1): solving $PW(i^*) = 0$ forces every intermediate balance, including a positive one, to grow at i^* itself, so the equation is asking one number to describe both what the project earns and what the firm does with early receipts. The MIRR splits those two jobs: the project's flows stay as they are, and the reinvestment is priced explicitly, outside the project. The cost of the repair is honesty about its inputs: the MIRR is not a property of the project alone. Change the stated external rates and the MIRR changes, so any quoted MIRR should come with its rates attached.

One caution closes the section. The MIRR is a repaired percentage, not a discovered one: it answers "what single rate summarizes this project, *given* what my other capital earns," which is a different and more modest question than the one the IRR answers for a simple investment. On graded work the expected routine is unchanged: classify by sign changes; if nonsimple, run the net-investment test; if mixed, decide by present worth at the MARR, and quote a MIRR (with its rates) when a percentage is demanded.

Lecture 18. This section is covered by Lecture 18.

Reference. Park, 3rd ed., Chapter 7, Section 7.4 (pp. 313–322).

Mutually exclusive alternatives. A set of options from which exactly one can be chosen. Selecting one rules out the others.

7.8 Why the Rate Cannot Rank Alternatives

The decision rule of Section 7.5 judges one project at a time. A new problem appears when alternatives are *mutually exclusive*: if project A earns 80% and project B earns 40%, and only one can be taken, is A the right choice? Surprisingly often, no. Ranking mutually exclusive alternatives by their own rates of return is a classic error. It appears in company memos, in published reports, and as a trap question on exams.

Example 7.9 — The ranking paradox

Two mutually exclusive one-year projects are available at a MARR of 10%. Project A invests 2,000 SAR and returns 3,600 SAR in one year. Project B invests 8,000 SAR and returns 11,200 SAR in one year. Rank them by IRR and by present worth, and state which ranking is correct.

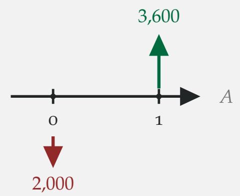

Figure 7.10. Project A: small investment, high percentage return.

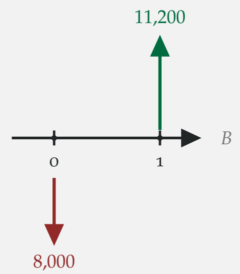

Figure 7.11. Project B: four times the investment, lower percentage return.

Solution.

Step 1: the rates.

Each project has one flow in and one flow out, so the rates follow by direct division:

$$i_A^* = \frac{3{,}600}{2{,}000} - 1 = 80\%, \qquad i_B^* = \frac{11{,}200}{8{,}000} - 1 = 40\%.$$

Ranking by rate says take A.

Step 2: the present worths at the MARR.

With $(P/F, 10\%, 1) = 0.9091$:

$$PW_A(10\%) = -2{,}000 + 3{,}600\,(0.9091) = +1{,}273 \text{ SAR},$$
$$PW_B(10\%) = -8{,}000 + 11{,}200\,(0.9091) = +2{,}182 \text{ SAR}.$$

Present worth says take $\boxed{\text{B}}$, and present worth is right, because the firm banks money, not percentages: 40% of 8,000 SAR builds more wealth than 80% of 2,000 SAR, as long as the larger investment still clears the hurdle.

Insight. The rate of return is a percentage, and a percentage is blind to scale. That blindness is the entire problem. If both projects could be taken, both would be, since both beat the MARR. Mutually exclusive means one must be chosen, and the choice between them is really a choice about the *extra* 6,000 SAR of capital that B requires. The right question is not ''which project has the higher rate,'' but ''is the extra money earning its keep?''

Incremental analysis. Rating the year-by-year difference between two alternatives (larger investment minus smaller) against the MARR, instead of comparing their individual rates.

7.9 *The Incremental Method*

Choosing B over A is the same as choosing A *plus* an extra investment in the difference B−A. The correction to the ranking paradox is therefore to compute the rate of return on that extra money.

Example 7.10 — Rating the increment

For the two projects of Example 7.9, form the incremental cash flow B−A, find its rate of return, and confirm the present-worth ranking.

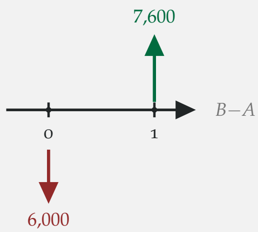

Figure 7.12. The increment B−A: the extra capital and the extra return, year by year.

Solution.

Subtract year by year: at $n = 0$, $-8{,}000 - (-2{,}000) = -6{,}000$; at $n = 1$, $11{,}200 - 3{,}600 = +7{,}600$. The increment has one sign change, so it is simple and its rate is trustworthy:

$$i^*_{B-A} = \frac{7{,}600}{6{,}000} - 1 = \boxed{26.67\%} > 10\% = \text{MARR} \Rightarrow \text{ choose B.}$$

The extra 6,000 SAR earns 26.67% inside project B. If the firm stopped at A, that money would go back to earning the MARR of 10%; keeping it out of B costs the firm money. The verdict now matches the present-worth ranking, as it always will when the increment is rated correctly (Figure 7.13).

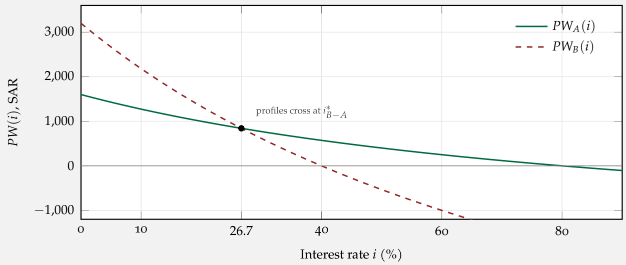

Figure 7.13. The two PW profiles cross exactly at the incremental rate, 26.67%. For any MARR below the crossing, B has the higher present worth; above it, A does. Each profile meets the axis at its own IRR (40% and 80%).

Insight. Why the incremental rate is the crossing point. Present worth is a sum over cash flows, so it subtracts term by term: $PW_B(i) - PW_A(i) = PW_{B-A}(i)$. The profiles are equal exactly where the increment's present worth is zero, which is the increment's i^*. This identity is also why incremental analysis always agrees with present worth: $PW_{B-A}(\text{MARR}) > 0$ means precisely $PW_B(\text{MARR}) > PW_A(\text{MARR})$.

The procedure, for two alternatives that are each acceptable on their own:

1. Order the alternatives by investment size, smaller first.

2. Form the increment (larger − smaller), year by year, and check its sign pattern.

3. If $i^*_\Delta \geq$ MARR, keep the **larger** investment; otherwise keep the **smaller**.

Two cautions. First, if the increment itself is nonsimple, fall back on present worth, exactly as in Section 7.6. Second, please note what the incremental rate is *not*: it is not the difference of the two rates. It is the rate of a difference. Those are different things.

Example 7.11 — Two wheel loaders

A mine must choose one of two wheel loaders. Both have 10-year lives, and the MARR is 15%. The data and the individual rates of return (computed by the method of Section 7.3) are in Table 7.3. Which loader should be chosen?

	L1	**L2**
Investment (year 0), SAR	24,000	36,000
Net annual saving (years 1–10), SAR	6,600	9,000
Salvage (year 10), SAR	2,400	6,000
IRR	24.8%	22.1%

Table 7.3. Data for Example 7.11.

Solution.

Step 1: screen each alternative.

Both rates clear the 15% MARR, so both are acceptable, and L1, the smaller investment, is the current choice. If one had failed this screen, the decision would already be over.

Step 2: form the increment L2−L1.

Subtract year by year: −12,000 at year 0; +2,400 per year in years 1–10; +3,600 of extra salvage at year 10. One sign change, so the increment is simple and its rate is trustworthy.

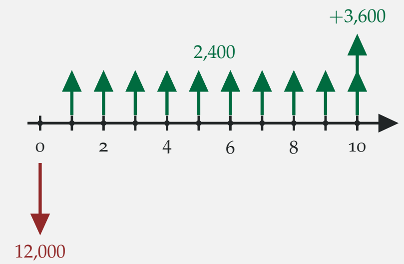

Figure 7.14. The increment L2−L1: extra capital now, extra savings each year, extra salvage at year 10.

Step 3: rate the increment.

Write $PW_\Delta(i) = -12{,}000 + 2{,}400\,(P/A,\,i,\,10) + 3{,}600\,(P/F,\,i,\,10)$ and bracket:

$$PW_\Delta(15\%) = -12{,}000 + 2{,}400\,(5.0188) + 3{,}600\,(0.2472) = +935.04 \text{ SAR},$$
$$PW_\Delta(18\%) = -12{,}000 + 2{,}400\,(4.4941) + 3{,}600\,(0.1911) = -526.20 \text{ SAR}.$$

Interpolate with Equation (7.2):

$$i^*_\Delta \approx 15\% + 3\% \times \frac{935.04}{935.04 + 526.20} = 15\% + 3\%\,(0.640) = \boxed{16.92\%}.$$

(The exact incremental rate is 16.85%.) Since $16.92\% > 15\% = \text{MARR}$, the extra 12,000 SAR earns its keep: $\boxed{\text{choose L2}}$, even though L1 has the higher individual rate.

Check with present worth.

At 15%, $PW_{L1} = -24{,}000 + 6{,}600\,(5.0188) + 2{,}400\,(0.2472) = +9{,}717.36$ SAR and $PW_{L2} = -36{,}000 + 9{,}000\,(5.0188) + 6{,}000\,(0.2472) = +10{,}652.40$ SAR. Present worth also ranks L2 first, as it must. Ranking by the individual rates would have picked the wrong loader.

In practice. The 2,400 SAR of extra annual saving is an engineering delta: fuel burn, cycle time, bucket payload, and availability, measured or estimated for each loader. The incremental rate is only as reliable as those measurements.

7.10 *Three or More Alternatives*

With three or more mutually exclusive alternatives, the same pairwise logic becomes a tournament.

Champion and challenger. In the pairwise tournament, the champion is the best alternative found so far; the challenger is the next larger investment that tries to displace it.

1. Order the alternatives by investment size: $D_1 < D_2 < D_3 < \dots$ (the do-nothing option, when allowed, enters at zero).
2. The *champion* is the smallest acceptable alternative.
3. Challenge it with the next larger one: rate the increment (challenger − champion).
4. If $i^*_\Delta \geq \text{MARR}$, the challenger becomes the new champion; otherwise the champion stays.
5. Repeat until no challengers remain. The last champion wins.

Every duel is pairwise, and every duel is incremental. At no point does anyone compare raw IRRs. The winner of this tournament is always the same alternative that maximizing present worth would pick, which is the point of the method. One practical note on the first duel: when do-nothing is allowed, the first challenge is the smallest alternative against do-nothing, and that duel is just the alternative's own rate against the MARR. So the acceptability screen is really the opening round of the tournament.

Example 7.12 — Three robotic palletizers

A packaging plant must choose one of three robotic palletizers. All have 8-year lives and no salvage value, and the MARR is 15%. The data are in Table 7.4. Run the champion–challenger tournament.

	R1	**R2**	**R3**
Investment (year 0), SAR	30,000	45,000	60,000
Net annual saving (years 1–8), SAR	7,000	10,800	13,800

Table 7.4. Data for Example 7.12.

Solution.

Step 1: order and screen.

The order by investment is R1, R2, R3. Each is a simple investment. Bracketing each own rate with $PW(15\%)$ and $PW(18\%)$, using $(P/A, 15\%, 8) = 4.4873$ and $(P/A, 18\%, 8) = 4.0776$:

$$\begin{aligned} \text{R1:} \quad & PW(15\%) = +1{,}411.10, \; PW(18\%) = -1{,}456.80 \;\Rightarrow\; i^* \approx 16.48\%, \\ \text{R2:} \quad & PW(15\%) = +3{,}462.84, \; PW(18\%) = -961.92 \;\Rightarrow\; i^* \approx 17.35\%, \\ \text{R3:} \quad & PW(15\%) = +1{,}924.74, \; PW(18\%) = -3{,}729.12 \;\Rightarrow\; i^* \approx 16.02\%. \end{aligned}$$

All three clear the 15% MARR, so all three stay in the tournament, and R1, the smallest, opens as champion.

Step 2: R2 challenges R1.

The increment R2−R1 is $-15{,}000$ now and $+3{,}800$ per year for 8 years; one sign change, simple.

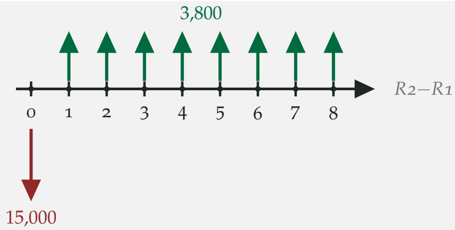

Figure 7.15. Duel 1: the increment R2−R1.

Bracket with $(P/A, 18\%, 8) = 4.0776$ and $(P/A, 20\%, 8) = 3.8372$:

$$PW_\Delta(18\%) = -15{,}000 + 3{,}800\,(4.0776) = +494.88 \text{ SAR},$$
$$PW_\Delta(20\%) = -15{,}000 + 3{,}800\,(3.8372) = -418.64 \text{ SAR},$$

$$i^*_\Delta \approx 18\% + 2\% \times \frac{494.88}{494.88 + 418.64} = \boxed{19.08\%} \geq 15\% \;\Rightarrow\; \text{R2 becomes champion.}$$

Step 3: R3 challenges R2.

The increment R3−R2 is −15,000 now and +3,000 per year for 8 years; one sign change, simple.

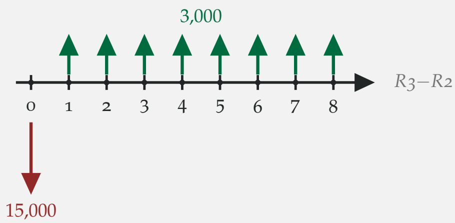

Figure 7.16. Duel 2: the increment R3−R2.

Bracket with $(P/A, 10\%, 8) = 5.3349$ and $(P/A, 12\%, 8) = 4.9676$:

$$PW_\Delta(10\%) = -15{,}000 + 3{,}000\,(5.3349) = +1{,}004.70 \text{ SAR},$$
$$PW_\Delta(12\%) = -15{,}000 + 3{,}000\,(4.9676) = -97.20 \text{ SAR},$$

$$i^*_\Delta \approx 10\% + 2\% \times \frac{1{,}004.70}{1{,}004.70 + 97.20} = \boxed{11.82\%} < 15\% \;\Rightarrow\; \text{R2 stays champion.}$$

Step 4: verdict and check.

No challengers remain, so $\boxed{\text{choose R2}}$. As a check, the present worths at the MARR are $PW_{R1} = +1{,}411.10$, $PW_{R2} = +3{,}462.84$, and $PW_{R3} = +1{,}924.74$ SAR: present worth also ranks R2 first. Please note that R2 wins even though it has neither the largest investment nor the smallest; the tournament settles each slice of capital on its own merits.

Service project. An alternative whose revenues are fixed or identical across the options, so only costs differ. Its own rate of return does not exist, but incremental rates between options do.

7.11 *Service Projects and Unequal Lives*

Two loose ends complete the method. The first is *service projects*. When the alternatives deliver the same service and only their costs differ, each alternative's cash flow is all outflows, with no sign change, so no individual rate of return exists (Section 7.4). The incremental method still works: the difference between a cheaper-to-run option and a cheaper-to-buy option typically has an investment-shaped pattern, extra capital now in exchange for cost savings later, and that increment can be rated against the MARR in the usual way. For service projects, the incremental rate is the only rate there is.

The second is *unequal lives*. An increment is a year-by-year subtraction, so both alternatives must be laid out over the same analysis period before subtracting. This is the same fairness rule as in chapter 5, and the same fixes apply: with repeatability, replicate to the lowest common multiple of the lives and subtract over that horizon; or compare annual equivalents over each life (chapter 6) when the repetition is identical; and a stated study period overrides everything. If two alternatives with different lives are subtracted year by year without a common horizon, the increment is meaningless, because one column of the subtraction runs out of service years. Set the horizon first; subtract second.

There is nothing new to memorize here. The horizon rules of chapter 5, the annual-equivalence shortcut of chapter 6, and this chapter's increments compose. Present worth judges a project in money at time zero, annual equivalence restates the judgment per year, and the rate of return restates it as a percentage, with the incremental method guarding the comparison step. Three languages, one criterion.

7.12 *Case Study from the Literature*

The internal rate of return is the most widely reported number in published techno-economic assessments: nearly every feasibility study of a mine, a refinery, a recycling plant, or a power project quotes one. Two open-access studies show the chapter's machinery, and the chapter's traps, operating at full industrial scale.

The first compares fourteen rare-earth mining projects with a single discounted-cash-flow model; the second evaluates a small plant that reclaims drinking water and saleable minerals from acid mine drainage. Both keep the currencies of the original papers, US dollars and South African rand respectively, since the numbers are theirs, not ours.

Reference. Jaroni, Friedrich, & Letmathe (2019); Shingwenyana, Shabalala, Mbhele, & Masindi (2021). Both open access; full references at the end of the chapter.

Example 7.13 — Case study: fourteen rare-earth projects, ranked by IRR and by NPV

Jaroni, Friedrich, and Letmathe (2019) built one discounted-cash-flow model, with common assumptions on ramp-up, cost structure, and prices, for fourteen advanced rare-earth mining projects outside China, and evaluated each by NPV and IRR against a minimum return of 10%. In their base case, the South African Steenkampskraal project earned the highest IRR of the fourteen, 25.8%, but, owing to its short 13-year life, generated an NPV of only about \$160 million; the European Norra Kärr project showed an NPV of −\$155 million with a payback of almost 16 years on a 20-year life. In their scenario with 25% higher prices for the critical rare-earth oxides, Steenkampskraal reached an IRR of about 33% with an NPV near \$300 million, while the much larger Australian Nolans project reached an NPV of about \$700 million at an IRR of only about 17%. Apply this chapter's rules to the published numbers.

Solution.

Step 1: the decision rule, project by project.

Screening is Section 7.5: at the study's 10% hurdle, base-case Steenkampskraal (25.8%) is comfortably acceptable, and Norra Kärr, whose base-case NPV is negative at 10%, is not — its IRR must sit below the hurdle, by Equation (7.3). The authors report exactly that: only seven of the fourteen projects cleared the 10% rate in the base case.

Step 2: the ranking paradox, at billion-dollar scale.

In the higher-price scenario, ranking by IRR puts Steenkampskraal (33%) far ahead of Nolans (17%); ranking by NPV reverses the order, \$700 million against \$300 million. This is Example 7.9 with nine more zeros: the percentage is blind to the fact that Nolans invests roughly \$1.2 billion and Steenkampskraal's shorter, smaller stream simply has less wealth in it. And the identity of Section 7.9 settles the choice without any new data: if an investor had to pick one,

$$PW_{\text{Nolans}} - PW_{\text{Steen}} = PW_{\Delta}(10\%) \approx 700 - 300 = +\$400 \text{ million} > 0,$$

so the incremental investment in the larger project clears the 10% hurdle, and the NPV ranking is the right one.

Step 3: internality, verified in the wild.

In a third scenario the authors replaced the uniform 10% rate with project-specific, risk-adjusted discount rates ranging from 7.5% to 24.5%. Every NPV moved, some by more than a billion dollars, and they note that "the IRRs of the projects did not change, as their calculation did not require an exogenously provided discount rate." That sentence is Equation (7.1) observed in field data: the rate is internal, and the hurdle it is compared against is where all the risk judgment lives. The published numbers behave exactly as the chapter predicts.

What did the researchers conclude? That economically attractive rare-earth projects exist outside China under every scenario tested; that the NPV method, not the IRR, was "primarily used to decide which project would be most profitable," because the IRR method alone is not suited to always leading to sound investment decisions — the same warning as Section 7.8, in their words; and that two projects, Mountain Pass and Bokan, cleared the hurdle under no scenario at all, a verdict supported by Mountain Pass's real-world closure in 2015 for unprofitability (Jaroni, Friedrich, & Letmathe, 2019).

Example 7.14 — Case study: an acid-mine-drainage reclamation plant

Shingwenyana, Shabalala, Mbhele, and Masindi (2021) evaluated a pilot-scale plant that treats acid mine drainage, the acidic wastewater of mining operations, and recovers drinking water, gypsum, and limestone for sale. Their financial model reports a capital cost of ZAR 452,000, a 10-year plant life, an IRR of 22% against a hurdle rate of 15%, and an NPV of ZAR 370,000 at that hurdle; the paper's conclusions quote an IRR of 26% with a payback of 3 years. The paper does not print the year-by-year cash flow row. Test what an outside reader *can* verify.

Solution.

Step 1: classify.

The plant is an investment followed by years of net sales revenue: one sign change, a simple investment, so a unique i^* exists and the IRR decision rule applies.

Step 2: the decision rule.

Both reported rates, 22% and 26%, sit far above the 15% hurdle, so the accept verdict is $\boxed{\text{robust to the discrepancy}}$ between the two figures — and a positive reported NPV at 15% is consistent with either, by Equation (7.3).

Step 3: reconstruct the cash flow the model must contain.

A uniform annual net cash flow A consistent with the reported 22% rate must satisfy $452{,}000 = A\,(P/A,\,22\%,\,10)$. With $(P/A,\,22\%,\,10) = 3.9232$,

$$A = \frac{452{,}000}{3.9232} = \boxed{\approx \text{ZAR } 115{,}200 \text{ per year}}.$$

That stream implies a simple payback of $452{,}000/115{,}200 = 3.9$ years and an NPV at 15% of

$$-452{,}000 + 115{,}213\,(5.0188) = +\text{ZAR } 126{,}200,$$

positive, the same verdict again. The reconstruction does not reproduce the paper's own NPV of ZAR 370,000 or its 3-year payback, which tells us the model's flows are not uniform (a ramp-up or escalation is likely inside it). An outside reader can verify the *decision*, not every digit — and noticing which is which is precisely the auditing skill this chapter's last section trains.

Insight. Note the division of labor in both case studies. Given the published rates, hurdle, life, and investment, the decision rule, the ranking identity, and the internality of i^* can all be checked from the outside with nothing but this chapter's tools; that part is arithmetic. What cannot be checked are the inputs themselves: ore grades, basket prices, recoveries, and cost estimates. A quoted IRR is only as good as the cash flows under it, and the flows are where the engineering lives.

7.13 Chapter Summary

- The **internal rate of return** i^* solves $PW(i^*) = 0$. It is the interest earned on the *unrecovered balance*: the project repays its capital and earns exactly i^*, with no surplus.
- **Direct solution**: a single future flow needs a root; a two-period project reduces to a quadratic in $x = (1 + i)^{-1}$ (keep the positive root).
- **Trial and error**: bracket a sign change of $PW(i)$ with the tables, then interpolate once with Equation (7.2). A spreadsheet checks the result with `=IRR`.
- **Count the sign changes** before trusting any rate: one change means simple, with a unique i^*; several changes mean nonsimple, with possibly multiple roots; no change means no rate of return exists.
- **Decision rule for simple investments**: accept when $i^* > \text{MARR}$. It always agrees with present worth, because $\text{MARR} < i^*$ is the same statement as $PW(\text{MARR}) > 0$.
- **Nonsimple investments**: decide by PW or AE at the MARR. The **net-investment test** (project balances at i^*, Equation (7.4)) separates *pure* investments, whose i^* still deserves trust, from *mixed* ones, whose i^* does not.
- For mixed investments, the **external rate of return (MIRR)** repairs the percentage: negatives discounted to year 0 at the finance rate, positives compounded to year N at the MARR, then one root, Equation (7.5). Accept when $\text{MIRR} \geq \text{MARR}$; with both external rates at the MARR, the verdict matches present worth exactly.
- **Never rank** mutually exclusive alternatives by their own IRRs: a percentage is blind to scale. Rate the **increment** (larger − smaller); if $i^*_\Delta \geq \text{MARR}$, take the larger investment.
- **Three or more alternatives**: run the champion–challenger tournament, pairwise and incremental, smallest first. The winner always matches the maximum-PW choice.

- **Service projects** have no individual rate; only increments can be rated. **Unequal lives** require a common horizon (LCM, AE, or a stated study period) before incrementing.

7.14 Exercises

These problems run from finding a single project's rate of return, through the sign-change scan and the multiple-root cases, to full incremental tournaments among mutually exclusive alternatives.

Finding and using the rate of return

Exercise 7.1. A packaging machine costs 25,000 SAR and has a salvage value of 5,000 SAR after 6 years. It saves 7,465 SAR per year in labor and costs 2,000 SAR per year to operate. The MARR is 15%. (a) Classify the investment as simple or nonsimple. (b) Find i^* by trial and error, bracketing with 10% and 15%. (c) State the accept/reject decision.

Exercise 7.2. A project's cash flows are $+4{,}000$, $+2{,}500$, and $+1{,}000$ SAR in years 0–2, with no negative flow anywhere. Its rate of return is:

(A) 0%

(B) Infinite: money for nothing

(C) Nonexistent: no sign change, so no rate of return is defined

(D) Equal to the MARR

A tailings-reprocessing project (in-class activity)

Exercise 7.3. *CopperISE307* is evaluating a 5-year tailings-reprocessing project that recovers residual copper from old mine waste. It costs 100,000 SAR today and returns a net 30,000 SAR per year in years 1–5. The firm's MARR is 12%. (a) Find the project's internal rate of return by trial and error: compute $PW(15\%)$ and $PW(20\%)$, interpolate, and state the accept/reject decision. (b) Before approval, regulators require the firm to fund a reclamation obligation of 90,000 SAR paid at closure (end of year 5), so the year-5 net flow becomes $30{,}000 - 90{,}000 = -60{,}000$ SAR. Count the sign changes of the revised series, classify it, explain why the plain "IRR vs. MARR" rule should not be trusted, draw the updated cash flow diagram, compute the present worth at the MARR, and make the correct decision.

i^* hunting, three ways (in-class activity)

Exercise 7.4. A project costs 18,000 SAR now and returns 8,000 SAR at year 1 and 13,000 SAR at year 2. (a) Find i^* exactly with the substitution $x = (1+i)^{-1}$ and the quadratic formula, to four decimals. (b) Find i^* again by evaluating PW at 9% and 12% and interpolating. How close is it to part (a)? (c) What does the spreadsheet formula `=IRR({-18000, 8000, 13000})` return? Do all three methods agree? (d) At a MARR of 10%, write the accept/reject decision sentence and verify it by the sign of $PW(10\%)$.

Classification clinic (in-class activity)

Exercise 7.5. (a) Classify each cash flow series below as S (simple), N (nonsimple), or X (no rate of return exists). Amounts are in SAR.

	$n = 0$	1	2	3	4
P1	-250	$+60$	$+970$	$+60$	$+60$
P2	-90	-100	-50	0	-20
P3	-300	$+220$	$+40$	-20	$+150$
P4	-70	$+20$	$+10$	$+40$	-20
P5	$-1{,}600$	$+1{,}500$	$+1{,}500$	-700	—
P6	$-12{,}000$	$+4{,}000$	$+4{,}500$	$+5{,}500$	—

(b) For P6, find i^* by bracketing (try 8% and 10%) and decide at a MARR of 9%. (c) Sketch the PW profiles of P1 and P5, marking the i^* crossings. What does P5's sign pattern warn you about? (d) Complete the sentence: for nonsimple flows, our decision tool is ____________________.

Incremental analysis

Exercise 7.6. A plant must pick one of two conveyor systems with 8-year lives; the MARR is 12%. C1 costs 45,000 SAR, nets 9,000 SAR per year, and salvages for

4,000 SAR at year 8. C2 costs 70,000 SAR, nets 13,500 SAR per year, and salvages for 7,000 SAR at year 8. Use incremental IRR analysis, bracketing the increment with 10% and 12%, to decide.

Exercise 7.7. Two mutually exclusive projects are both simple and both acceptable: $\text{IRR}_A = 30\%$ and $\text{IRR}_B = 20\%$, with a MARR of 12%, and B is the larger investment. The incremental rate is $i^*_{B-A} = 15\%$. Which project should be chosen?

(A) A: it has the highest IRR

(B) B: the increment earns $15\% > 12\%$ MARR

(C) A: the increment earns less than IRR_A

(D) Neither: the IRRs disagree

The incremental tournament (in-class activity)

Exercise 7.8. Three dryers are available for a revenue project; do-nothing is allowed. All have 5-year lives and no salvage, and the MARR is 12%. D1 costs 10,000 SAR and nets 3,300 SAR per year; D2 costs 16,000 SAR and nets 5,000 SAR per year; D3 costs 25,000 SAR and nets 7,300 SAR per year. (a) Order by investment and show that D1's own IRR clears the MARR. (b) Let D2 challenge D1: form the increment, find its IRR, and give the verdict. (c) Let D3 challenge the winner: increment, IRR, verdict, and the champion. (d) Verify by computing $PW(12\%)$ for all three and confirming that the tournament winner has the maximum present worth. (e) Write the decision sentence.

Exercise 7.9. Alternative A costs 14,000 SAR and nets 4,600 SAR per year for 6 years. Alternative B costs 21,000 SAR and nets 6,500 SAR per year for 6 years. The MARR is 12%. Form the increment, bracket i^*_Δ, state the choice, and self-check with present worth.

The net-investment test and the external rate of return

Exercise 7.10. *Najd Copper* is offered a two-year mobile-crushing contract at a satellite pit. Mobilizing the crusher costs 100,000 SAR now; the contract pays a net 255,000 SAR at the end of year 1; demobilization and site restoration cost 161,000 SAR at the end of year 2. The MARR is 12%, and the firm sets both external rates equal to the MARR. (a) Classify the series. (b) Find every i^* exactly with the substitution $x = (1+i)^{-1}$. (c) Run the net-investment test at each root and classify the investment as pure or mixed. (d) Compute the MIRR and state the decision. (e) Verify the decision with the sign of $PW(12\%)$.

Exam-style problems

The problems below follow the format, length, and point weights of Major Exam 2. Work each part in exam order: draw the diagram, classify the investment, write the factor notation, bracket, interpolate, box the answer, and close with the decision sentence.

Exercise 7.11. *Najd Copper* evaluates a thickener upgrade at its concentrator. The upgrade costs 150,000 SAR now, saves a net 36,000 SAR per year for 6 years, and the recovered internals can be sold for 18,000 SAR at the end of year 6. The MARR is 10%. (a) (*5 pts*) Classify the investment as simple or nonsimple, and state what the classification guarantees about i^*. (b) (*12 pts*) Find i^* by trial and error, bracketing with 12% and 15%, and interpolate once. (c) (*5 pts*) State the accept/reject decision and verify it with the sign of $PW(10\%)$. (d) (*3 pts*) Management debates the hurdle. For each MARR of 8%, 12%, and 15%, state accept or reject.

Exercise 7.12. *Wadi Phosphate* must choose one of two vibrating screens for its beneficiation plant. Both have 8-year lives, and the MARR is 12%. S1 costs 80,000 SAR, nets 21,000 SAR per year, and salvages for 8,000 SAR. S2 costs 120,000 SAR, nets 29,500 SAR per year, and salvages for 12,000 SAR. (a) (*6 pts*) Show that each screen is individually acceptable using the sign of its present worth at the MARR. (b) (*12 pts*) Form the increment S2−S1, classify it, find its rate by bracketing with 12% and 15%, and state which screen to buy. (c) (*7 pts*) The individual rates are about 21.0% for S1 and 18.9% for S2. Explain in one or two sentences why ranking by these rates would mislead here, and confirm your choice with the present-worth ranking.

Exercise 7.13. *Wadi Lithium* considers recommissioning an old evaporation pond. The works cost 200,000 SAR now, the pond then yields a net 500,000 SAR at the end of year 1, and closure and reclamation cost 312,000 SAR at the end of year 2. The MARR is 15%. (a) (*4 pts*) Classify the series and state the maximum number of rates it may have. (b) (*10 pts*) Find every i^* exactly, using the substitution $x = (1+i)^{-1}$ and the quadratic formula. (c) (*7 pts*) Make the correct accept/reject decision at the MARR. (d) (*4 pts*) Both roots in part (b) exceed the MARR, yet your decision in part (c) is what it is. Explain in one or two sentences.

Assessing outside recommendations

Each problem below quotes a finished analysis produced by someone else: an AI assistant, a vendor, or a consultant. The work looks professional and its arithmetic is internally consistent, but each analysis contains one or two genuine method errors. Decide whether to accept or reject the recommendation, name each flaw precisely, and produce the corrected number.

Exercise 7.14. *Najd Copper* must choose one of two mutually exclusive haul-road upgrade designs. Both last 5 years with no salvage, and the MARR is 10%. Design K costs 15,000 SAR and saves 6,000 SAR per year; design M costs 60,000 SAR and saves 19,500 SAR per year. A consultant's memo reads:

> "Design K: $(P/A, i, 5) = 15{,}000/6{,}000 = 2.5$, so $\text{IRR}_K \approx 28.6\%$. Design M: $(P/A, i, 5) = 60{,}000/19{,}500 = 3.0769$, so $\text{IRR}_M \approx 18.7\%$. Both designs clear the 10% hurdle comfortably. Design K delivers the higher return on investment, and we recommend K."

Accept or reject the memo? Name the flaw, produce the corrected analysis, and state the right design.

Exercise 7.15. *Wadi Phosphate* asks an AI assistant to evaluate a pit-dewatering contract: buy and mobilize pumps for 80,000 SAR now, receive a net contract payment of 250,000 SAR at the end of year 1, then pay 180,000 SAR for demobilization and pit restoration at the end of year 2. The MARR is 12%. The assistant reports:

> "I solved $PW(i^*) = 0$ for the cash flows $(-80{,}000, +250{,}000, -180{,}000)$ and obtained $i^* = 12.5\%$. Since $12.5\% > 12\%$, the contract clears the hurdle. The margin is thin but positive. Recommendation: accept."

Accept or reject the assistant's work? Name the flaw, find what the assistant missed, and produce the corrected decision.

References

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