3 Money Management: Nominal and Effective Rates, Loans, and Financing

After this chapter you will be able to:

1. distinguish the nominal annual rate r from the effective annual rate i_a, and state the role of the compounding frequency M
2. compute the effective annual rate $i_a = (1 + r/M)^M - 1$, and its continuous-compounding limit $i_a = e^r - 1$
3. convert a rate quote into the effective rate per payment period, $i = (1 + r/M)^C - 1$ with $C = M/K$, when payments and compounding do not match
4. compute the payment $A = P(A/P, i, N)$ of an amortized loan, its remaining balance B_t, and the interest and principal parts of each payment
5. diagnose an add-on loan quote, and compute the effective rate the borrower really pays
6. decide whether to refinance a loan by comparing the saving on the remaining payments with the refinancing fee

Chapter 2 built the interest factors on two quiet assumptions: interest compounds exactly once per year, and the rate quoted to us is the rate we actually pay. In practice, neither is usually true. A bank says "9% compounded monthly," and that short phrase changes the true cost of the loan, because interest is charged and added to the balance twelve times a year, not once. This chapter closes that gap. It first decodes the language of rate quotes, separating the nominal rate from the compounding frequency and computing the effective rate that is truly paid or earned. It then handles the realistic case where the payment schedule and the compounding schedule do not line up, using one master move: the effective rate per payment period. With the right rate in hand, the chapter turns to the most widely used application, the amortized loan: the payment, the remaining balance, the split of each payment into interest and principal, and the full repayment schedule. Two commercial situations follow, the add-on loan, which quotes one rate and charges another, and the refinancing decision, which replaces the remaining payments of an old loan with a new one. The chapter closes with a first look at where investment money comes from, debt and equity, as a bridge to the cost of capital in chapter 12. The next chapter (chapter 4) adds the second force acting on money over time: inflation.

How much a quoted rate hides.

Q1. A bank quotes "12% compounded monthly." What rate is charged each month, and is the true annual cost more or less than 12%?

Q2. Two banks quote the same nominal 10%. One compounds annually, the other daily. Do the two loans cost the same?

Q3. You borrow 1,000 SAR at 10% per year and repay with two equal year-end payments of 576.19 SAR. After the first payment, do you owe exactly half the loan?

Answers.

Q1. The monthly charge is $12\%/12 = 1\%$ on the unpaid balance. The true annual cost is *more* than 12%: interest is charged on interest twelve times, so the effective annual rate is $(1.01)^{12} - 1 = 12.68\%$.

Q2. *No.* Annual compounding costs exactly 10.00% per year. Daily compounding costs $(1 + 0.10/365)^{365} - 1 = 10.52\%$ per year. At the same nominal rate, more frequent compounding always costs the borrower more.

Q3. *No.* During year 1 the debt grows to $1{,}000\,(1.10) = 1{,}100$; the payment removes 576.19, leaving 523.81 SAR, more than half. The first payment must cover 100 SAR of interest before it touches the principal. Early payments are always interest-heavy.

Lecture 8. This section is covered by Lecture 8.

Reference. Park, 3rd ed., Sections 3.1–3.2 (pp. 114–121).

Nominal interest rate (APR). The quoted annual rate r, also called the annual percentage rate. It must be read together with the compounding frequency.

Compounding frequency. The number of times per year, M, that interest is computed and added to the balance.

3.1 *Nominal and Effective Interest Rates*

A rate quote has two distinct parts, and most of the confusion in this chapter comes from treating them as one. The quote "r compounded M times a year" means that the rate charged *each period* is

$$i = \frac{r}{M},$$

so "18% compounded monthly" charges $18\%/12 = 1.5\%$ each month on the unpaid balance. The quoted 18% is the *nominal* rate, the APR. It is a label, not the true yearly cost, because within the year each month's interest joins the balance and itself earns interest.

Effective annual rate (APY). The one rate i_a that truly represents the interest paid or earned in a year, once all within-year compounding is included. Also called the annual percentage yield.

The *effective annual rate* compounds the per-period rate across the whole year:

$$i_a = \left(1 + \frac{r}{M}\right)^M - 1. \tag{3.1}$$

When $M = 1$, Equation (3.1) reduces to $i_a = r$: with annual compounding there is no within-year compounding to add anything, and the nominal and effective rates agree. That is the special case Chapter 2 lived in. For any $M > 1$, the effective rate sits above the nominal rate, and it rises as M rises. Table 3.1 shows how far, for a fixed nominal 9%.

Compounding M	annual 1	semiannual 2	quarterly 4	monthly 12	weekly 52	daily 365
i_a for $r = 9\%$	9.000%	9.203%	9.308%	9.381%	9.409%	9.416%

Table 3.1. The effective annual rate of a nominal 9% under increasingly frequent compounding. The effective rate rises with M, but by less and less each step.

Insight. A quote of "9% compounded monthly" is *not* 9%. The APR and the APY answer different questions: the APR names the per-period charge (r/M per period), while the APY states the cumulative cost of a full year of those charges. A bank advertising what you will *earn* displays the larger APY; a bank advertising what you will *pay* displays the smaller APR. Regulation requires both to be disclosed precisely because the two numbers differ.

Example 3.1 — The true cost of a supplier credit line

A workshop finances spare parts on a supplier credit line quoted at *18% APR, billed monthly*. What is the effective annual rate, and what does a 10,000 SAR balance carried for a full year actually cost?

Solution.

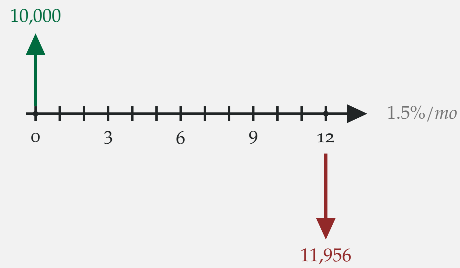

Figure 3.1. The credit line over twelve months: 10,000 SAR received now, one balance repaid a year later after twelve monthly compoundings.

The two parts of the quote are $r = 18\%$ and $M = 12$, so the monthly charge is $18\%/12 = 1.5\%$. Compound it across the year:

$$i_a = \left(1 + \frac{0.18}{12}\right)^{12} - 1 = (1.015)^{12} - 1 = \boxed{19.56\%}.$$

A 10,000 SAR balance carried for one year therefore costs $0.1956 \times 10{,}000 =$ $\boxed{1{,}956 \text{ SAR}}$, not the 1,800 SAR that the headline 18% suggests. The extra 156 SAR is the price of monthly compounding: each month's interest joins the balance and earns interest itself for the rest of the year.

Continuous compounding. The limit of ever more frequent compounding: $M \to \infty$, giving $i_a = e^r - 1$.

3.2 *Continuous Compounding*

Table 3.1 shows the effective rate rising with M, but by smaller and smaller steps. Push the frequency to its limit, compounding every instant, and the expression does not blow up; it settles to a specific value:

$$i_a \;=\; \lim_{M\to\infty} \left(1 + \frac{r}{M}\right)^M - 1 \;=\; e^r - 1. \tag{3.2}$$

For a nominal 9%, continuous compounding gives $e^{0.09} - 1 = 9.417\%$, barely above the daily figure of 9.416%.

Insight. Equation (3.2) is a ceiling. No matter how aggressively a lender compounds, the effective annual rate can never climb past $e^r - 1$. This also gives a quick sanity band: any effective rate you compute must land between the nominal rate r (the

$M = 1$ floor) and $e^r - 1$ (the continuous ceiling). An answer outside that band signals an arithmetic slip.

Effective rates exist to make different quotes comparable. Never compare nominal rates across different compounding frequencies; convert every offer to its effective annual rate first, then rank.

Example 3.2 — Ranking three financing quotes

A battery-materials supplier needs a 100,000 SAR working-capital loan for one year and collects three quotes: **Bank A**, 9.0% compounded quarterly; **Bank B**, 8.8% compounded monthly; **Bank C**, 8.7% compounded continuously. Which bank offers the cheapest money?

Solution.

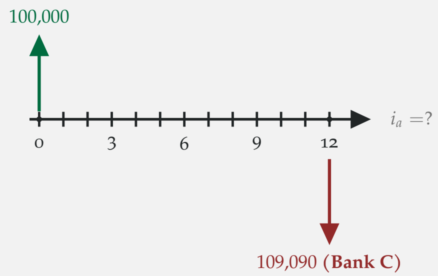

Figure 3.2. One year of the loan under the cheapest quote: 100,000 SAR in at month 0, one repayment a year later. Each bank's effective annual rate sets the height of the repayment arrow.

Convert each quote to an effective annual rate with Equations (3.1) and (3.2):

$$
\begin{aligned}
i_a^A &= (1 + 0.09/4)^4 - 1 = (1.0225)^4 - 1 = 9.308\%, \\
i_a^B &= (1 + 0.088/12)^{12} - 1 = 9.164\%, \\
i_a^C &= e^{0.087} - 1 = \boxed{9.090\%}.
\end{aligned}
$$

Rank by effective rate: C (9.090%) $<$ B (9.164%) $<$ A (9.308%). Decision sentence: borrow from **Bank C**; despite the most aggressive compounding in the list, its lower nominal rate gives the lowest effective annual cost, 9.09% against 9.16% and 9.31%. On the 100,000 SAR loan the gap between the best and worst quote is about 218 SAR in one year. Please note what decided the ranking: not the nominal rates (which would rank C best anyway here), and not the compounding frequencies alone, but the combination, read through i_a.

Reference. Park, 3rd ed., Section 3.2 (pp. 118–121).

3.3 *The Effective Rate per Payment Period*

Real transactions are messier still: you pay on one schedule while the bank compounds on another. A firm pays quarterly while the account compounds monthly; a fund deposits weekly into an account quoted with continuous compounding. Every interest factor of Chapter 2 demands a rate that matches the payment period exactly, so the task is to build a single rate aligned with when the payments actually happen.

K, M, C. K = payments per year; M = compoundings per year; $C = M/K$ = compoundings inside one payment period.

Count three numbers. K is the number of payments per year. M is the number of compoundings per year. Their ratio $C = M/K$ counts the compoundings that sit inside one payment period. The *effective rate per payment period* compounds the per-period charge r/M across those C steps:

$$i = \left(1 + \frac{r}{M}\right)^C - 1, \qquad C = \frac{M}{K}, \tag{3.3}$$

and under continuous compounding the same idea gives

$$i = e^{r/K} - 1. \tag{3.4}$$

Once this rate is in hand, every factor from Chapter 2 works unchanged, with N counted in *payments*.

Insight. The rate and the period must always describe the same unit of time. If payments are quarterly, the factor needs a quarterly rate and N counts quarters; twelve quarterly payments over three years means $N = 12$, never $N = 36$. Mixing a monthly rate with a count of quarters, or an annual rate with a count of months, is the single most common error in this chapter's exam problems.

Example 3.3 — A quarterly maintenance fund with monthly compounding

A maintenance fund saves 5,000 SAR at the end of every quarter for 3 years, in an account that pays 8% compounded monthly. Find the balance immediately after the last deposit.

Solution.

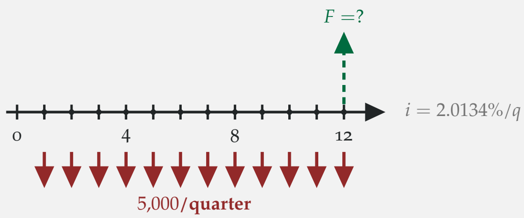

Figure 3.3. Twelve quarterly deposits over three years; the axis counts quarters, and the rate is the effective rate per quarter.

Step 1: align the rate with the payments.

Payments are quarterly ($K = 4$) and compounding is monthly ($M = 12$), so $C = 3$ monthly compoundings sit inside each quarter. By Equation (3.3),

$$i = \left(1 + \frac{0.08}{12}\right)^3 - 1 = 2.0134\% \text{ per quarter.}$$

Check the direction: the naive $8\%/4 = 2\%$ ignores the compounding inside the quarter, so the true rate must sit slightly above 2%, and it does.

Step 2: apply the ordinary series factor.

Three years of quarterly deposits is $N = 12$ payments:

$$F = 5{,}000\,(F/A,\ 2.0134\%,\ 12) = 5{,}000\,(13.4222) = \boxed{67{,}111 \text{ SAR}}.$$

Please note the discipline: N counts the 12 payments, not the 36 months, and the rate is the quarterly rate that matches them.

Example 3.4 — Annual payments under weekly compounding

A 10,000 SAR equipment loan is repaid with 5 equal *annual* payments, but the bank compounds *weekly* at a nominal 5%. Find the annual payment.

Solution.

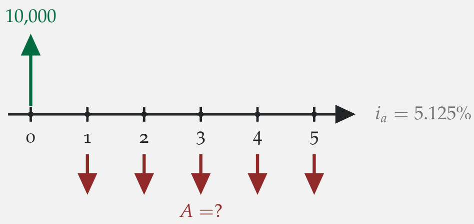

Figure 3.4. The loan received at year 0 and five equal annual repayments, valued at the effective annual rate.

Step 1: effective rate per payment period.

The payment period is the year, so $K = 1$, $M = 52$, and $C = 52$:

$$i = \left(1 + \frac{0.05}{52}\right)^{52} - 1 = 5.125\% \text{ per year.}$$

All the weekly compounding is now folded into one annual rate.

Step 2: capital recovery at the matched rate.

$$A = 10{,}000\,(A/P,\, 5.125\%,\, 5) = 10{,}000\,(0.23177) = \boxed{2{,}318 \text{ SAR per year}}.$$

When the schedules do not match, the first move is always Equation (3.3); only after the rate and the payments agree does a factor enter.

Lecture 9. This section is covered by Lecture 9.

Reference. Park, 3rd ed., Section 3.4 (pp. 128–139).

Amortized loan. A loan repaid in equal periodic payments, each of which covers the interest accrued on the outstanding balance and retires part of the principal.

Remaining balance. B_t, the amount still owed immediately after payment t: the present worth of the $N - t$ payments not yet made.

3.4 *Amortized Loans*

The single most used application of this chapter is the amortized loan: borrow P, repay with N equal payments at rate *i per payment period*. Three quantities describe everything about it.

First, the payment itself is a capital-recovery calculation from Chapter 2:

$$A \;=\; P\,(A/P,\, i,\, N). \tag{3.5}$$

Second, the balance still owed at any moment is the present worth of the payments that have not yet been made:

$$B_t \;=\; A\,(P/A,\, i,\, N - t). \tag{3.6}$$

Third, each payment splits into two jobs. It first pays the interest accrued on what was owed, and whatever remains retires principal:

$$I_t = i\,B_{t-1}, \qquad PP_t = A - I_t. \tag{3.7}$$

Early in the loan the balance is large, so most of each payment is interest; late in the loan the balance is small, so most of each payment is principal.

Insight. Equation (3.6) is the one idea that answers every "balance after k payments" question: the remaining balance is *always* the value today of the payments still to come. There is no need to march through the loan payment by payment. The same bookkeeping returns in chapter 5 with the roles reversed: there the *project* owes *you*, and the year-by-year account is called the project balance.

Example 3.5— Financing a lithium flotation cell line

A plant finances a 280,000 SAR lithium flotation cell line at 9% compounded monthly, repaid over 60 equal monthly payments. Find (a) the monthly payment, (b) the interest/principal split of the first payment, (c) the balance right after the 24th payment, and (d) the split of the 25th payment.

Solution.

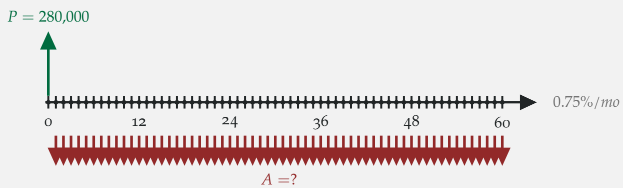

Figure 3.5. The loan: 280,000 SAR received at month 0 and sixty equal monthly repayments.

Step 1: the payment.

Payments and compounding are both monthly, so the periods match: $i = 9\%/12 = 0.75\%$ per month and $N = 60$:

$$A = 280{,}000\,(A/P,\,0.75\%,\,60) = 280{,}000\,(0.020758) = \boxed{5{,}812.34\ \text{SAR/mo}}.$$

Step 2: split of the first payment.

Before any payment, the balance is the full $B_0 = 280{,}000$:

$$I_1 = 0.0075\,(280{,}000) = 2{,}100, \qquad PP_1 = 5{,}812.34 - 2{,}100 = \boxed{3{,}712.34 \text{ SAR}}.$$

More than a third of the first payment is interest.

Step 3: balance after the 24th payment.

Thirty-six payments remain, so value them today:

$$B_{24} = 5{,}812.34\,(P/A,\, 0.75\%,\, 36) = 5{,}812.34\,(31.4468) = \boxed{182{,}780 \text{ SAR}}.$$

We did not walk through 24 payments one by one; we valued the 36 still to come.

Step 4: split of the 25th payment.

$$I_{25} = 0.0075\,(182{,}780) = 1{,}370.85, \qquad PP_{25} = 5{,}812.34 - 1{,}370.85 = \boxed{4{,}441.49 \text{ SAR}}.$$

Two years in, the interest share of a payment has fallen from 36% to 24%, because the balance has come down. Table 3.2 shows how the pattern completes: by the final payments, almost everything is principal.

Month t	Payment A	Interest I_t	Principal PP_t	Balance B_t
0				280,000.00
1	5,812.34	2,100.00	3,712.34	276,287.66
2	5,812.34	2,072.16	3,740.18	272,547.48
3	5,812.34	2,044.11	3,768.23	268,779.25
⋮	⋮	⋮	⋮	⋮
59	5,812.34	86.21	5,726.13	5,769.07
60	5,812.34	43.27	5,769.07	0.00

Table 3.2. The amortization schedule of the flotation-line loan (Example 3.5). Each row applies Equation (3.7): interest on the previous balance first, principal with the remainder. The interest column shrinks as the balance falls.

A shorter loan shows the whole pattern at once. Figure 3.6 draws all five payments of the conveyor loan of Example 2.16: watch the interest slice fall while the bar it sits in never changes height.

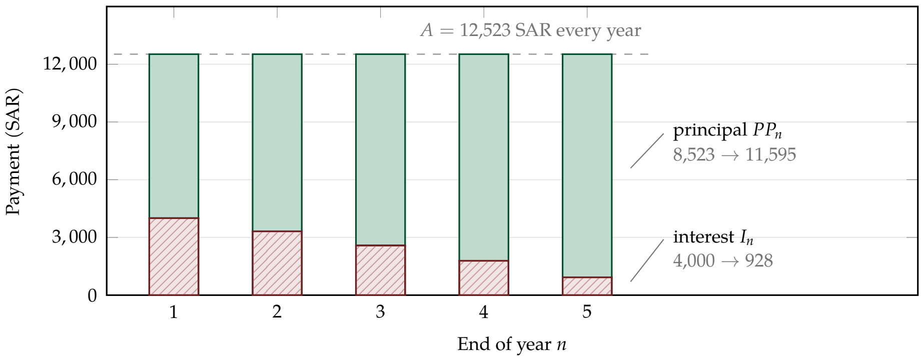

Figure 3.6. The amortized conveyor loan of Example 2.16 paid out year by year. Every bar is the same 12,523 SAR payment, but the interest slice shrinks from 4,000 to 928 SAR as the balance falls, and the principal slice grows to fill what it leaves. Nothing about the payment changes; only the job it does.

In practice. Loan tenor should not outlive the asset. Financing a 60-month loan against a machine with a 48-month service life leaves a year of payments on equipment that is already retired; the schedule of Table 3.2 must be read next to the maintenance plan's estimate of the machine's life.

3.5 *Add-On Loans and the True Cost of a Quote*

Not every installment loan is amortized. In an *add-on* loan, the lender computes interest on the full original principal for the full term, even as the principal is paid down. The quoted add-on rate is therefore not comparable to an amortized rate, and the gap is large.

Add-on loan. A loan whose interest is computed on the *full* original principal for the *whole* term, then added on; principal plus interest is split into equal payments.

Example 3.6 — A 6% add-on retrofit offer

A vendor offers a 50,000 SAR haul-truck retrofit at "just 6% add-on, 36 months." Find the monthly payment and the true effective annual rate of the offer.

Solution.

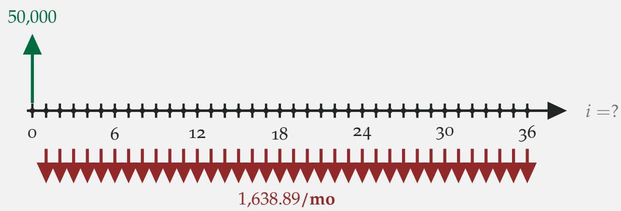

Figure 3.7. The add-on offer as the borrower experiences it: 50,000 SAR in, thirty-six equal payments out. The rate that makes the two sides equivalent is the true cost.

Step 1: the add-on arithmetic.

Interest is charged on the full 50,000 SAR for all three years:

$$\text{owed} = 50{,}000\,(1 + 0.06 \times 3) = 59{,}000\,\text{SAR}, \qquad A = \frac{59{,}000}{36} = 1{,}638.89\,\text{SAR/mo}.$$

Step 2: the true rate.

The true cost is the rate i that makes 50,000 SAR today equivalent to that payment stream:

$$(P/A,\, i,\, 36) = \frac{50{,}000}{1{,}638.89} = 30.5085.$$

i/mo	$(P/A, i, 36)$
0.75%	31.4468
i	**30.5085**
1.00%	30.1075

Table 3.3. Bracketing the unknown monthly rate of Example 3.6: the target factor 30.5085 falls between the 0.75% and 1% rows.

The factor falls between $i = 0.75\%$ (31.4468) and $i = 1\%$ (30.1075); see Table 3.3. Interpolating,

$$i \approx 0.75\% + 0.25\% \times \frac{31.4468 - 30.5085}{31.4468 - 30.1075} = 0.925\% \text{ per month}$$

(the exact solution is 0.9235%). The effective annual rate is

$$i_a = (1.009235)^{12} - 1 = \boxed{11.66\%}.$$

The "6%" add-on actually costs about 11.7% per year, nearly double the quoted figure. Decision sentence: treat the add-on quote as an 11.66% effective loan and compare it against amortized offers on that basis, never against their nominal rates.

Insight. Why roughly double? In an amortized loan, interest is charged only on what is still owed, and the balance falls with every payment. The add-on structure keeps charging interest on principal that has already been returned. Halfway through the term you owe about half the principal but still pay interest on all of it, so the effective charge per SAR actually borrowed is close to twice the quoted rate. This is exactly why regulators require lenders to disclose effective rates.

In practice. Revenue-based financing is quoted the same way: ''repay 1.4 times the advance over four years'' sounds like 40%, but as an effective annual rate on the declining balance it is closer to 18–20%, and the exact figure depends on how fast the revenue arrives. Solve for the equivalence rate before comparing it with a bank quote.

3.6 *Mortgages and the Refinancing Decision*

A mortgage is an amortized loan with a long tenor, and everything in Section 3.4 applies to it directly. One decision is so common that it deserves its own example: interest rates have fallen since the loan was signed, and a bank offers to *refinance*, replacing the remaining payments with a new, cheaper schedule, for a fee. The remaining balance, Equation (3.6), is exactly the amount being refinanced, and the comparison is between two payment streams and a fee, valued at the borrower's opportunity rate.

Refinancing. Paying off the remaining balance of an existing loan with a new loan at a new rate, usually for a fee.

In practice. Vendors often bundle the machine price and the financing into a single monthly figure. The engineering comparison of two machines requires unbundling: value the payment stream at your own opportunity rate to recover the effective machine price, then compare machines and financing separately.

Example 3.7 — Refinancing a warehouse loan

Three years ago a firm financed a warehouse with a 400,000 SAR loan at 9% compounded monthly over 120 monthly payments. Rates have fallen: a bank now offers to refinance the remaining balance at 6% compounded monthly over the remaining 84 months, for a 4,000 SAR processing fee paid now. The firm's short-term funds earn 6% compounded monthly. Should it refinance?

Solution.

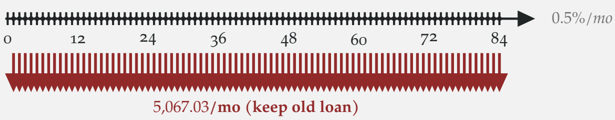

Figure 3.8. Keeping the old loan: 84 remaining payments of 5,067.03 SAR, valued at the firm's 0.5% monthly opportunity rate.

Step 1: the old payment and the balance being refinanced.

$$A_{\text{old}} = 400{,}000\,(A/P,\,0.75\%,\,120) = 400{,}000\,(0.012668) = 5{,}067.03\,\text{SAR/mo}.$$

After 36 payments, 84 remain:

$$B_{36} = 5{,}067.03\,(P/A,\,0.75\%,\,84) = 5{,}067.03\,(62.1540) = 314{,}936 \text{ SAR}.$$

Step 2: the new payment.

The new loan repays exactly B_{36} at $6\%/12 = 0.5\%$ per month over 84 months:

$$A_{\text{new}} = 314{,}936\,(A/P,\,0.5\%,\,84) = 314{,}936\,(0.014609) = 4{,}600.76 \text{ SAR/mo}.$$

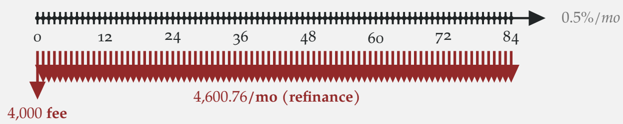

Figure 3.9. Refinancing: the fee now, then 84 smaller payments.

Step 3: value the saving at the opportunity rate.

Refinancing saves $5{,}067.03 - 4{,}600.76 = 466.27$ SAR per month for 84 months, and costs 4,000 SAR now. At the firm's 0.5% monthly opportunity rate,

$$\text{PW} = 466.27\,(P/A,\,0.5\%,\,84) - 4{,}000 = 466.27\,(68.4530) - 4{,}000 = \boxed{+27{,}918 \text{ SAR}}.$$

Decision sentence: refinance; the fee is recovered many times over, and the firm is better off by about 27,918 SAR in present-worth terms.

Debt financing. Raising investment money by borrowing: banks and bondholders receive a contractual interest rate, priced by the tools of this chapter.

Equity financing. Raising investment money from owners: no contractual rate, but the owners expect the return available elsewhere at similar risk.

3.7 *Debt, Equity, and the Cost of Financing*

Everything in this chapter priced one source of money: debt. A lender quotes a rate, and Equations (3.1) through (3.7) translate the quote into payments and balances. But firms also finance investments with *equity*, the owners' money. Equity carries no invoice and no amortization schedule, yet it is not free: the owners expect the return their money could earn elsewhere at similar risk, and a firm that persistently pays them less loses them.

A project is usually financed by a blend of the two, and the blended price of the firm's money, the weighted average cost of capital, is one of the inputs behind the MARR used from chapter 5 onward. That calculation, including the tax treatment of interest, is developed in chapter 12 (Section 12.7). For now, keep the division of labor: this chapter prices a *loan* exactly; chapter 12 prices the firm's *pool of money* approximately, as a weighted average in which the loan rate is one ingredient.

> **Insight.** The loan tools of this chapter and the project tools of later chapters meet in one sentence: a project financed at a blended rate must earn at least that rate, or someone who supplied the money is not being paid. The effective annual rate is the honest unit in which the debt share of that blend is measured.

3.8 Case Study from the Literature

The rates in this chapter's examples were quoted by fictional banks. What do real lenders charge real projects? For renewable-energy projects the question is hard to answer, because most are financed by private bank loans whose terms are trade secrets. Steffen (2020) addressed this by systematically reviewing 19 published studies that estimated the cost of capital, the blended debt-and-equity rate of Section 3.7, for solar and wind projects, harmonizing them into after-tax figures for 46 countries over 2009–2017. The paper's equation for the ''vanilla'' weighted average cost of capital is exactly the blend previewed above: the debt share times the cost of debt plus the equity share times the expected return on equity. Because the studies span years with different benchmark rates, the paper states each country's result as a markup on the interbank benchmark rate (LIBOR) and also evaluates it at 2017 benchmark levels. All rates are effective annual rates in nominal terms, benchmarked in the currencies of the underlying studies; the figures quoted below are the paper's harmonized percentages, not amounts in any single currency.

Reference. Steffen (2020), *Energy Economics*, 88:104783. Open access (CC BY); full reference at the end of the chapter.

Three of the paper's findings matter for this chapter. First, the financing rate is a large slice of the cost of renewable electricity: the paper cites evidence that the cost of capital makes up 12–37% of the levelized cost of solar electricity in Germany, and up to about half in developing countries. Second, the rank order across technologies is consistent everywhere: solar photovoltaic projects borrow most cheaply, onshore wind next, offshore wind highest. Third, the differences across countries are far larger than the differences across banks in any one country: the reported after-tax cost of capital for solar projects, evaluated at average 2017 benchmark rates, is 4.2% for Saudi Arabia (a markup of 2.5 points over the benchmark), 2.7% for Germany, 5.4% on average across OECD countries, and 10.6% for India.

Example 3.8 — Case study: what the financing rate does to a solar project

Using the after-tax cost-of-capital figures reported by Steffen (2020) at 2017 benchmark rates, 4.2% for Saudi Arabia, 5.4% for the OECD average, and 10.6% for India, compute the equal annual capital charge on each 10 million USD of solar investment recovered over a 25-year project life, and compare.

Solution.

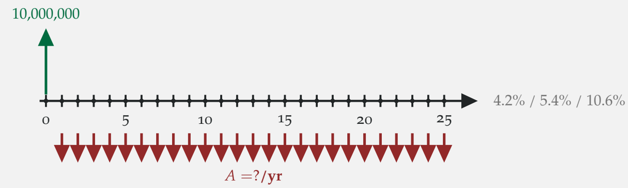

Figure 3.10. Each 10 million USD of solar capital, recovered as an equal annual charge over 25 years. The financing rate sets the height of the arrows.

The annual capital charge is a capital-recovery calculation at each country's rate:

$$\begin{aligned}
A_{\text{Saudi}} &= 10{,}000{,}000\,(A/P,\ 4.2\%,\ 25) \\
&= 10{,}000{,}000\,(0.06537) = \boxed{653{,}700\ \text{USD/yr}}, \\
A_{\text{OECD}} &= 10{,}000{,}000\,(A/P,\ 5.4\%,\ 25) \\
&= 10{,}000{,}000\,(0.07382) = \boxed{738{,}200\ \text{USD/yr}}, \\
A_{\text{India}} &= 10{,}000{,}000\,(A/P,\ 10.6\%,\ 25) \\
&= 10{,}000{,}000\,(0.11529) = \boxed{1{,}152{,}900\ \text{USD/yr}}.
\end{aligned}$$

The same panels, the same sun, the same 10 million USD of hardware: at India's reported financing rate the annual capital charge is 76% higher than at Saudi Arabia's. Since a solar plant's cost is nearly all capital, this ratio passes almost directly into the cost per kilowatt-hour, which is why the paper argues that financing terms belong beside hardware prices in any comparison of where renewable projects are cheap.

Insight. The case study also shows why this book keeps insisting on effective annual rates. Steffen (2020) could only harmonize 19 studies into one table because every entry was first converted to the same unit, an effective annual after-tax rate. Quotes stated on different compounding bases, or before and after tax, cannot be averaged or ranked until they are converted; the paper's method is Example 3.2 applied at the scale of a literature.

Steffen (2020) closes with a warning that also fits this chapter: using one standard discount rate for every country and technology misprices projects, because real lenders do not charge one rate. The same warning, scaled down, applies to a firm comparing two banks: the quote is not the cost; the effective rate is.

3.9 *Chapter Summary*

- A rate quote has **two parts**: the nominal annual rate r (the APR) and the compounding frequency M. The rate charged per period is r/M.
- The **effective annual rate** $i_a = (1 + r/M)^M - 1$ (the APY) is the true yearly cost. Offers are compared *only* through effective rates, never through nominal rates quoted on different compounding bases.
- **Continuous compounding** is the limit $i_a = e^r - 1$; it is the ceiling that compounding frequency can reach. Every effective rate lies between r and $e^r - 1$.
- When payments and compounding differ, build the **effective rate per payment period**: $i = (1 + r/M)^C - 1$ with $C = M/K$, or $i = e^{r/K} - 1$ under continuous compounding. Then every Chapter 2 factor applies with N counted in *payments*.
- **Amortized loans**: payment $A = P\,(A/P, i, N)$; remaining balance $B_t = A\,(P/A, i, N - t)$, the present worth of the payments still to come; split $I_t = i\,B_{t-1}$ and $PP_t = A - I_t$. Early payments are interest-heavy, late payments principal-heavy.
- **Add-on loans** charge interest on the full principal for the whole term. Their true effective cost is roughly twice the quoted add-on rate; recover it by solving $(P/A, i, N) = P/A_{\text{payment}}$ for i.

- **Refinancing** replaces the remaining payments (whose principal is B_t) with a new schedule for a fee; value the monthly saving against the fee at the borrower's opportunity rate.
- Projects are financed by **debt and equity**. This chapter prices debt exactly; the blended cost of the two, the WACC behind the MARR, is developed in chapter 12.

3.10 *Exercises*

The problems begin with converting nominal quotes into effective rates, then handle payment periods that do not match the compounding period, and close with full loan work: amortization schedules, add-on quotes, and refinancing.

Effective rates

Exercise 3.1. A supplier quotes a nominal 12% per year. Compute the effective annual rate under annual, semiannual, quarterly, monthly, and daily compounding, and under continuous compounding. What pattern do the six numbers show?

Exercise 3.2. A factory makes *quarterly* deposits into a machine-replacement fund that earns 9% compounded *monthly*. (a) Find the effective rate per quarter, showing the (K, M, C) values used. (b) If the factory deposits 6,000 SAR at the end of every quarter for 5 years, find the balance immediately after the last deposit. (c) Instead, the factory wants the fund to reach exactly 200,000 SAR after the same 5 years; find the required equal quarterly deposit.

Loan analysis

Exercise 3.3. A factory borrows 300,000 SAR for a production machine at 9% compounded monthly, repaid in 48 equal monthly installments. (a) Find the monthly payment. (b) Split the first payment into interest and principal. (c) Find the remaining balance immediately after the 24th payment, using the present-worth method rather than a table.

Exercise 3.4. A supplier offers the same 300,000 SAR as Exercise 3.3, but as a *6% add-on loan* over the same 48 months. (a) Find the monthly payment under the add-on quote. (b) Find the true effective monthly and annual rates it charges. (c) Which financing is cheaper, the supplier's "6%" or the bank's 9% compounded monthly, and why is the naive comparison wrong?

Exercise 3.5. An investment fund earns 9% compounded quarterly. Approximately how long will it take a deposit to *triple*?

(A) About 8.0 years.
(B) About 12.3 years.
(C) About 12.7 years.
(D) About 22.2 years.

Exam-style problems

The following problems are written in the format of the Major Exam: a scenario, lettered parts with point values, and full work expected. Align the rate with the payment period before any factor, write the factor notation before substituting numbers, box each final answer with its units, and end with a decision sentence.

Exercise 3.6. Financing a concentrate dryer at Wadi Phosphate. The firm borrows 180,000 SAR for a concentrate dryer at **8% compounded quarterly**, repaid with **20 equal quarterly payments** over 5 years. (a) (*4 pts*) State the effective rate per payment period and the effective annual rate of this loan. (b) (*8 pts*) Find the quarterly payment. (c) (*8 pts*) Find the remaining balance immediately after the 8th payment, and split the 9th payment into interest and principal. (d) (*4 pts*) The site accountant notes that the firm will pay $20 \times A$ in total and calls the difference over 180,000 SAR "the price of the loan." Explain in two sentences why that total is not an equivalent-cost measure, and state the loan's true annual cost.

Exercise 3.7. Cash discount or vendor financing at Najd Copper. A crusher can be bought two ways: pay **230,000 SAR cash** now (a discounted price), or pay the list price of **240,000 SAR financed at 3.6% APR compounded monthly** with **36 equal monthly payments** and nothing down. The firm's treasury funds earn **6% compounded monthly**. (a) (*6 pts*) Find the monthly payment under the financing option. (b) (*6 pts*) State which interest rate must be used to compare the two options, and why. (c) (*8 pts*) Compute the present worth of the financing option at that rate and state the decision. (d) (*4 pts*) Treasury conditions change and the firm's funds now earn 12% compounded monthly. Recompute the comparison and state whether the decision changes.

Assessing outside recommendations

The two problems below present finished analyses produced by others. In each, the arithmetic shown is correct; the flaw is in the method. For each problem, (a) decide whether to accept the recommendation, (b) identify each error or omission precisely, and (c) produce the corrected numbers.

Exercise 3.8. An AI ranks two working-capital lines. Najd Copper asks an AI assistant to rank two one-year credit lines. It replies:

> *Bank X quotes 8.0% compounded quarterly, so you pay 8.0% per year. Bank Y quotes 8.1% compounded annually, so you pay 8.1% per year. Since* $8.0 < 8.1$*, Bank X is cheaper. Recommendation: borrow from Bank X.*

(a) (*4 pts*) Should the firm accept the recommendation? (b) (*8 pts*) Identify the error precisely. (c) (*8 pts*) Produce the corrected comparison and the correct choice.

Exercise 3.9. A bank memo compares loans by total payments. Wadi Phosphate needs 100,000 SAR of equipment financing and receives an internal memo:

> *Offer L1: 10% compounded monthly, 48 monthly payments of 2,536.26 SAR; total paid* $=$ $121{,}740$ *SAR. Offer L2: 14% compounded monthly, 24 monthly payments of 4,801.29 SAR; total paid* $=$ $115{,}231$ *SAR. L2 costs 6,509 SAR less in total. Recommendation: take L2.*

(a) (*4 pts*) Should the firm accept the recommendation? (b) (*8 pts*) Identify each error or omission precisely. (c) (*8 pts*) Produce the corrected comparison and state the correct basis for the choice.

References

Steffen, B. (2020). Estimating the cost of capital for renewable energy projects. *Energy Economics*, 88, 104783. `https://doi.org/10.1016/j.eneco.2020.104783`

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