2 *The Time Value of Money*

After this chapter you will be able to:

1. explain why money has a time value, using earning power and purchasing power
2. build a cash flow diagram that follows the end-of-period convention and a single point of view
3. distinguish simple interest, $F = P(1 + iN)$, from compound interest, $F = P(1 + i)^N$, and state when two cash flows are economically equivalent
4. compute F or P for a single amount with $(F/P, i, N)$ and $(P/F, i, N)$, reading each factor from the interest factor table
5. compute the worth of a uniform series with (F/A), (A/F), (A/P), and (P/A), including a deferred annuity and a perpetuity
6. compute the present worth of a composite stream by separating it into single amounts, uniform series, and gradients, using $(P/G, i, N)$ for a linear gradient

This chapter is the foundation of the course. Almost every later method, including present-worth analysis, rate-of-return analysis, the comparison of alternatives, and replacement decisions, is built directly on the ideas in this chapter. The chapter is organized in the following order: why money has a time value; the elements of an interest transaction; the cash flow diagram; simple and compound interest; economic equivalence; the single-payment factors; the uniform-series factors; and the gradient factors. The chapter also explains how to read the interest factor tables, because choosing the correct factor and reading its value correctly is an important part of every problem.

Three intuitions about waiting for money.

Q1. Other things equal, would you rather receive 1,000 SAR today or 1,000 SAR in two years? Why?

Q2. A bank offers 8% compounded annually. If you deposit 100 SAR today, how much is in the account after one year?

Q3. At a positive interest rate, is 100 SAR in two years *equivalent* to 100 SAR in three years?

Answers.

Q1. *Today.* Money received now can be invested to grow; money delayed two years cannot earn anything in the meantime. The two amounts are identical in face value but not in economic value.

Q2. $100(1.08) = 108$ SAR. After one period of compounding, the balance is the original deposit multiplied by $(1 + i)$.

Q3. *No.* Equivalence requires equal value at a common date for a given rate. Even though the face amounts are equal, the year-2 amount has had an extra year of earning power compared with the year-3 amount, so they are not economically equal.

Lecture 3. This section is covered by Lecture 3.

Time value of money. The same amount of money has more value when it is received sooner, because it can be invested to earn a return.

Earning power. Money held now can be invested to earn a return, so its amount increases over time.

Purchasing power. The quantity of goods that a fixed amount of money can buy. It usually decreases over time because of inflation.

2.1 *Why Money Has a Time Value*

A sum of money is described by two things, not one: the amount, and the time at which it arrives. The same amount can have more or less value, depending only on its timing. There are two reasons.

- The first reason is *earning power*. Money held today can be invested, lent, or deposited, and it then earns a return and increases. Money that is not received until later cannot begin to earn until later.
- The second reason is *purchasing power*. The quantity of goods that a fixed amount of money can buy changes over time. It usually decreases because of inflation.

Example 2.1 — Time value of money in action

For each of the three pairs of cash flows below, state which one is preferable and explain *why*. The third pair is harder than the first two; identify what makes it harder and what additional piece of information would be needed to settle it.

1. 200 SAR today vs. 150 SAR today.
2. 100 SAR today vs. 100 SAR in two years.
3. 200 SAR today vs. 225 SAR in two years.

Solution.

- **(1)** *200 SAR today.* Both arrive at the same date, so timing is not a factor; the larger amount wins.
- **(2)** *100 SAR today.* The amounts are equal in face value but the later one cannot earn anything in the meantime. At any positive interest rate, the present amount is worth more.
- **(3)** *It depends on the interest rate.* The later amount is larger, but it is delayed by two years. We need a method to move both cash flows to a common date before we can compare them. That method is the time value of money, with the interest rate as its converter. The remainder of this chapter develops it.

Interest. The cost of money: the amount a borrower pays, and a lender earns, for the use of money over time.

The method to compare amounts at different dates is the *interest rate*, which is a percentage applied to an amount of money over a stated period. This is one of the most important tools in engineering economics that we will use throughout the course to compare amounts at different dates.

2.2 *Interest: The Cost of Money*

Interest is the cost of money. For the borrower, it is a cost: the amount paid for using another person's money. For the lender, the same interest is an income: the reward for waiting. It is one flow of money, viewed from two sides. One way to understand it is to compare it with rent. When a bank lends money, it rents that money to the borrower, and interest is the rent.

Market interest rate. The rate used throughout this course that already reflects both earning power and the effect of inflation, so the cash amounts in a problem can be used directly, without adjustment.

The rate used throughout this course is the *market interest rate*, which already reflects both earning power and the effect of inflation, so the cash amounts in a problem can be used directly, without adjustment.

2.3 *The Elements of an Interest Transaction*

Present amount. the initial sum, the amount of money at time zero. It is also called the *principal*.

Interest rate. the cost of money per period, written as decimal value. For example, 8% is written as 0.08.

Number of periods. The duration counted in interest periods.

Period cash flow. An equal amount received or paid each period. This is also called the *annuity* or the *equal end-of-period amount*.

Future amount. The result after interest acts for N periods. This is also called the *compound amount*.

Five symbols describe any transaction that involves interest. They appear in every problem in the course.

- P, *present amount*: the initial sum, the amount of money at time zero. Note that P is conventionally always at time zero. However, do not confuse this with the use of P in formula notation, where P is the present value of a future amount F, for which it might not be at time zero.

- i, *interest rate*: the cost of money per period. We will always use the market interest rate, which already reflects both earning power and the effect of inflation. In chapter 3, the chapter on money management, we will use other interest rates, such as the nominal interest rate and the effective interest rate, which are defined and explained there. We will sometime refer to this also as discount rate, when the rate is used to discount a future amount to its present value. In casual reading, it might also be called the value of money, which, is rather subjective and might have different values for different people.

- N, *number of periods*: the duration, counted in interest periods. This is generally the number of periods we move transaction forward or backward in time. Generally, it will have positive value when we move the transaction forward in time, and negative value when we move it backward in time for formula notation. However, when using table factor notation, N is the number of periods in the table, and it will always have positive value but requires appropriate factor notation to be used.

- A, *period cash flow*: an equal amount received or paid each period. This is the most common cash flow in engineering economics, and it is used to model the cash flows of a loan, an investment, a lease, a bond, a stock, a project, a business, a government program, a natural resource, etc.

- F, *future amount*: the result after interest acts on the money. It represents the future value of cash flows after its earning power is taken into account for N periods.

2.4 *The Cash Flow Diagram (CFD)*

The cash flow diagram is the most important tool in the course, and it should be the first step in almost every problem. The horizontal axis is time, divided into interest periods labeled $0, 1, \ldots, N$. Time zero is the present. On this axis we draw arrows for the cash flows: an up arrow is an inflow (money received) and a down arrow is an outflow (money paid), as in Figure 2.1.

Three conventions apply. First, **arrow lengths are approximate**: a longer arrow indicates a larger amount, but the arrows are not drawn exactly to scale. Second, **one point of view only**: a diagram is drawn either from the borrower's point of view or from the lender's point of view. The same transaction seen from the other side has every arrow reversed. Third, the **end-of-period convention**: any cash flow that occurs during a period is placed at the end of that period.

When working with a problem, draw the cash flow diagram first. Most errors in this course are timing errors, and the diagram makes the timing visible. Once the diagram is correct, the choice of factor and the calculation usually follow without difficulty.

Cash flow diagram. A time axis $0, 1, \ldots, N$ representing cash flows.

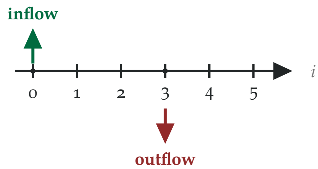

Figure 2.1. Inflows point up, outflows point down.

2.5 *Two Ways to Calculate Interest*

Simple interest. Interest charged only on the original principal. The growth is linear, $F = P(1 + iN)$.

Simple interest charges the rate only on the original principal. Interest that has already been earned does not itself earn interest, so the interest amount is the same in every period and the growth is linear:

$$F = P(1 + iN). \tag{2.1}$$

Compound interest. Interest charged on principal plus accumulated interest; growth is exponential, $F = P(1 + i)^N$.

Compound interest charges the rate on the principal plus any interest already accumulated. Earned interest joins the principal and earns interest of its own, so the growth is exponential:

$$F = P(1 + i)^N. \tag{2.2}$$

Insight. Why the exponent appears. After one period, the balance is $P(1 + i)$. In the second period, the rate applies to this new balance, which gives $P(1 + i)(1 + i) = P(1 + i)^2$. Each period multiplies the current balance by the same factor $(1 + i)$, so after N periods this factor has been applied N times, giving $F = P(1 + i)^N$. The exponent is therefore a count of how many times the balance is multiplied. Under simple interest, the earned interest is never added back to the balance, so the same amount iP is added N times, and N appears as a multiplier rather than an exponent. This is the difference between linear growth and exponential growth.

Example 2.2 — Simple vs. compound interest

Consider a loan of 1,000 SAR at 8% for three years (Figure 2.2). Find the future amount under simple interest and under compound interest, and compare the difference.

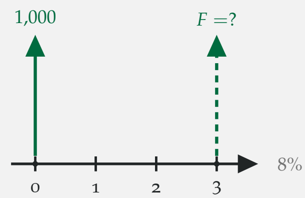

Figure 2.2. A single present amount of 1,000 SAR grown for 3 years; compute F under each interest rule.

Solution.

For 1,000 SAR at 8% over three years, simple interest gives $1{,}000(1+0.24) = 1{,}240$ SAR, while compound interest gives $1{,}000(1.08)^3 = 1{,}259.71$ SAR.

The difference of 19.71 SAR is small here, but it becomes larger as N increases, because exponential growth eventually exceeds linear growth. Unless a problem states otherwise, *all interest in this course is compound*.

Example 2.3 — When does simple interest equal compound interest?

Suppose you deposit P today at rate i. For which periods N is the future amount F the same under simple interest and under compound interest?

Solution.

Using (2.1) and (2.2), set the two future amounts equal:

$$P(1+iN) = P(1+i)^N \quad \Longrightarrow \quad 1+iN = (1+i)^N.$$

This is one equation, but N appears both inside a product and as an exponent, so it cannot be rearranged into a closed form $N = \ldots$ by algebra. Instead we check the equation directly.

At $N = 0$: both sides equal 1. At $N = 1$: both sides equal $1 + i$. So the two methods always agree at $N = 0$ and $N = 1$.

For any N strictly between 0 and 1, the compound side is smaller, and for any $N > 1$ the compound side is larger. To see this, compare term by term using the binomial expansion for an integer $N \geq 2$:

$$(1+i)^N = 1 + Ni + \binom{N}{2} i^2 + \cdots + i^N = (1 + iN) + \underbrace{\binom{N}{2} i^2 + \cdots + i^N}_{>0 \text{ for } i>0,\ N\geq 2}.$$

Every extra term is positive, so $(1+i)^N > 1 + iN$ for all $N \geq 2$. These extra terms are exactly the interest earned on previously earned interest, which simple interest never adds.

Conclusion. With a positive rate, simple and compound interest give the same future amount only at $\boxed{N = 0 \text{ and } N = 1}$. For every $N > 1$, compound interest gives strictly more, and the gap widens as N grows (Figure 2.3).

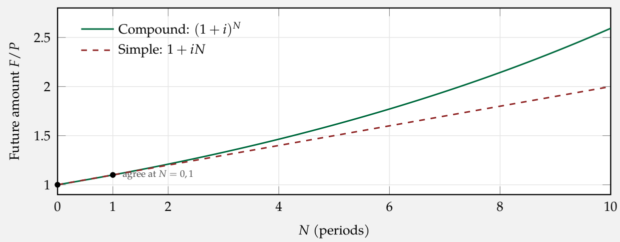

Figure 2.3. Simple and compound future amounts at $i = 10\%$. The two curves cross only at $N = 0$ and $N = 1$; for every later period the compound curve is strictly above, and the gap grows with N.

Example 2.4 — Doubling time at the same nominal rate

A robotics fund holds 50,000 SAR. **Bank A** pays 6% simple interest; **Bank B** pays 5% compounded annually. For each bank, find how many years it takes the fund to *double*.

Solution.

Bank A (simple). Doubling means $F = 2P$, so $1 + iN = 2$, giving

$$N_A = \frac{1}{i} = \frac{1}{0.06} = \boxed{16.67 \text{ years}}.$$

Bank B (compound). Set $(1+i)^N = 2$ and take logarithms:

$$N_B = \frac{\ln 2}{\ln 1.05} = \frac{0.6931}{0.04879} = \boxed{14.21 \text{ years}}.$$

Bank B doubles the fund in fewer years even though its headline rate (5%) is *lower* than Bank A's (6%). This is the basic moral of compounding: reinvesting the earned interest beats a higher rate that never compounds. The Rule of 72 gives $72/5 = 14.4$ years for Bank B, close to the exact 14.21.

Lecture 4. This section is covered by Lecture 4.

Economic equivalence. Two cash flows are equivalent at rate i if they have the same value at a common date. Equivalence means the holder is indifferent between them.

2.6 *Economic Equivalence*

Two cash flows, or two complete series of cash flows, are *economically equivalent* if they have the same value at a common point in time, for a given interest rate. If two amounts are equivalent, one can be exchanged for the other with no gain and no loss, so the holder is *indifferent* between them.

Insight. Equivalence depends on a specific interest rate, because the rate is what converts an amount at one date into its value at another date. Two series with different timing are moved forward or backward by different factors. When the rate changes, these factors change by different amounts, so two series that were equal at one rate are usually not equal at another rate. An equivalence is a statement about one rate only. It is not permanent.

Example 2.5 — Two repayment plans are equivalent

A firm borrows 300,000 SAR at 9% to install a processing line. Two repayment plans are on the table, shown in Figures 2.4 and 2.5. **Plan 1** repays the loan in five equal year-end payments. **Plan 2** pays nothing for four years and then makes a single payment at year 5. Show that the bank is indifferent between the two plans.

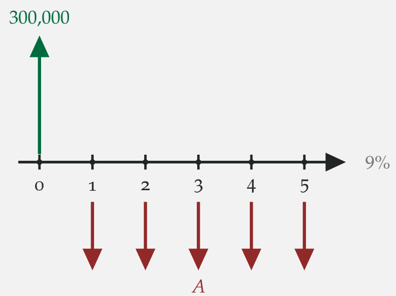

Figure 2.4. Plan 1: loan received now, repaid in five equal end-of-year installments.

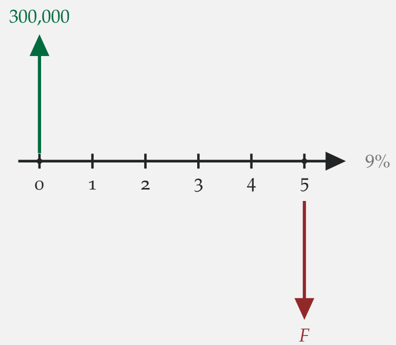

Figure 2.5. Plan 2: loan received now, single repayment at year 5.

Solution.

If we add the payments, the two plans are not equal: Plan 1 totals about 385,600 SAR and Plan 2 totals about 461,600 SAR. However, the bank is willing to offer either plan and is indifferent between them, because at 9% both plans have the same present worth of 300,000 SAR. Plan 2 pays more in total only because it pays later, and the additional amount is exactly the interest that compensates the bank for waiting.

Example 2.6 — Finding the equivalence rate

A vendor offers 15,000 SAR today; an alternative offer is 20,073 SAR five years from now. At what interest rate are the two offers *economically equivalent*?

Solution.
Set the present and future amounts equal at the unknown rate i:

$$15{,}000\,(1+i)^5 = 20{,}073 \quad\Longrightarrow\quad (1+i)^5 = \frac{20{,}073}{15{,}000} = 1.3382.$$

Take the fifth root:

$$1+i = (1.3382)^{1/5} = 1.0600 \quad\Longrightarrow\quad i = \boxed{6.0\%}.$$

The two offers are equivalent for an investor whose money is worth 6%; if the investor's rate is *above* 6%, the cash-now offer is better, and if *below* 6%, the later 20,073 SAR is better.

Example 2.7 — Lump vs. installments: sensitivity to rate

A foundation offers either 60,000 SAR now (Plan L) or 8,000 SAR per year for 12 years (Plan I). Find the present worth of Plan I at $i = 6\%$, $i = 8\%$, and $i = 10\%$, and state which plan wins at each rate.

Solution.
Plan L is fixed at 60,000 SAR. Plan I is a uniform series; use (P/A):

$$P_I(i) = 8{,}000\,(P/A,\, i,\, 12).$$

Reading the factor table:

$$\begin{aligned}
i = 6\%: &\quad P_I = 8{,}000\,(8.3838) = \boxed{67{,}071 \text{ SAR}} > 60{,}000 \Rightarrow \text{installments win.}\\
i = 8\%: &\quad P_I = 8{,}000\,(7.5361) = \boxed{60{,}289 \text{ SAR}} \approx 60{,}000 \Rightarrow \text{indifferent.}\\
i = 10\%: &\quad P_I = 8{,}000\,(6.8137) = \boxed{54{,}510 \text{ SAR}} < 60{,}000 \Rightarrow \text{lump sum wins.}
\end{aligned}$$

The break-even rate sits near 8%. The takeaway is that the equivalence of two plans depends entirely on the rate: a plan that is preferable at one rate can flip to inferior at a slightly higher one, because future payments are discounted more sharply.

Lecture 4. This section is covered by Lecture 4.

2.7 *The Single-Payment Factors*

We now define the two factors that move a single amount through time.

2.7.1 *Compound amount factor* $(F/P, i, N)$

This factor answers the question: given P today, what is its value after N periods at rate i? The answer is Equation (2.2), written with a standard notation:

$$F = P(1+i)^N = P\,(F/P, i, N). \tag{2.3}$$

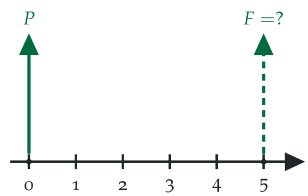

Figure 2.6. (F/P) moves a present amount forward to a future amount.

$(F/P, i, N)$**.** Read as "find F given P." It moves P forward N periods. Its value is $(1+i)^N$.

The notation can be read as a phrase. The letter written first is what you want to find, and the letter written second is what you already have, so (F/P) means "find F given P."

2.7.2 *Present worth factor* $(P/F, i, N)$

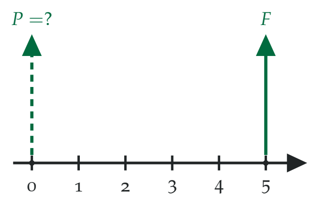

Figure 2.7. (P/F) moves a future amount back to the present.

$(P/F, i, N)$**.** Read as "find P given F." It moves F back N periods. Its value is $(1+i)^{-N}$.

This factor is the reverse of the first one: given a future amount F, what is its value today? Finding the value today of a future amount is called *discounting*.

$$P = F(1+i)^{-N} = F\,(P/F, i, N). \tag{2.4}$$

Insight. The present-worth factor is the reverse of growth. To undo $F = P(1+i)^N$, divide both sides by $(1+i)^N$, which is the same as multiplying by $(1+i)^{-N}$. This is why the only change is the sign of the exponent, and why the two factors are exact reciprocals: $(F/P, i, N)\,(P/F, i, N) = 1$ on every row of the table. Either factor can be used to check the other, so you never need to look up both.

Example 2.8 — Discounting a future amount

What present amount P is indifferent to receiving 150,000 SAR in seven years, if money is worth 6%?

Solution. This is a direct (P/F) application.

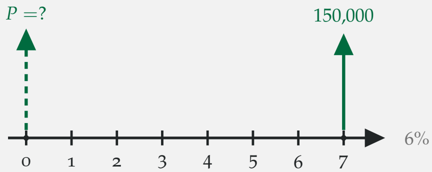

Figure 2.8. A single future amount discounted to the present.

$$P = 150{,}000\,(P/F, 6\%, 7) = 150{,}000\,(0.6651) = \boxed{99{,}765 \text{ SAR}}.$$

The present amount is about two-thirds of the future amount; the missing third is the interest that money compounded over seven years at 6% would have earned.

Reference. Park, 3rd ed., Appendix B (interest factor tables); this book, Appendix B.

2.8 *How to Read the Interest Factor Table*

The factor values are listed in tables, with one page for each interest rate and one row for each value of N. You do not need to compute $(1+i)^N$ by hand each time; you read it from the table. The procedure has four steps.

1. Find the page for your interest rate.
2. Go down to the row for your N.
3. Read across to the column for the factor you need: (F/P) to move an amount forward, (P/F) to move an amount back, and so on.
4. Multiply that factor by your known cash flow.

N	(F/P)	(P/F)
6	1.5007	0.6663
7	1.6058	0.6227
8	1.7182	0.5820
9	1.8385	0.5439

Table 2.1. Excerpt from the $i = 7\%$ page. The highlighted row, $N = 8$, gives $(F/P) = 1.7182$ and $(P/F) = 0.5820$. Note that $1.7182 \times 0.5820 \approx 1$.

Table 2.1 shows an excerpt from the 7% page. To move a present amount forward eight years, find the $N = 8$ row and read the (F/P) column: 1.7182. To move a future amount back eight years, read the (P/F) column on the same row:

0.5820. The two factors on any row are reciprocals, which is a quick way to find a reading error.

Example 2.9— Using the interest factor table

A robotics research fund of 100,000 SAR is invested at 7% for 8 years. Find the future amount.

Solution.

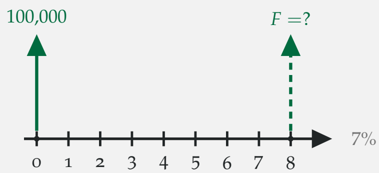

Figure 2.9. One deposit at year 0; the unknown F at year 8.

Step 1: factor notation. $F = P\,(F/P, 7\%, 8)$.

N	(F/P)	(P/F)
6	1.5007	0.6663
7	1.6058	0.6227
8	**1.7182**	0.5820
9	1.8385	0.5439

Table 2.2. Excerpt from the $i = 7\%$ interest factor table used for Example 2.9.

Step 2: read the table. Open the $i = 7\%$ page and look down to the $N = 8$ row, then across to the (F/P) column (see Table 2.2 in the margin). The cell reads 1.7182.

Step 3: substitute.

$$F = 100{,}000\,(1.7182) = \boxed{171{,}820 \text{ SAR}}.$$

The exact formula gives $100{,}000(1.07)^8 = 171{,}818.62$ SAR, which differs by less than two SAR. The table is rounded to four decimals, and that is precise enough for any engineering economic decision.

2.9 *Solving for the Rate or the Number of Periods*

Equation (2.2) contains four quantities: P, F, i, and N. If any three are known, the fourth can be found. Dividing by P isolates the growth term $(1+i)^N = F/P$, and the unknown is then either the exponent or the base.

Insight. A logarithm finds an unknown exponent; a root finds an unknown base. To find N, take the logarithm of both sides of $(1+i)^N = F/P$. The exponent becomes a multiplier, which gives $N = \ln(F/P)/\ln(1+i)$. To find i, take the Nth root of both sides: $1+i = (F/P)^{1/N}$, so $i = (F/P)^{1/N} - 1$.

Rule of 72. Years to double $\approx 72/(\text{rate in percent})$. It is accurate for rates of about 4% to 15%.

For a quick check of doubling time, the *Rule of 72* estimates the number of years to double as 72 divided by the rate written as a plain percentage. At 12%, that is $72/12 = 6$ years, close to the exact $\ln 2/\ln 1.12 = 6.12$.

Example 2.10 — Deriving the Rule of 72

Derive the Rule of 72! Show that it is accurate for rates of about 4% to 15%.

Solution.

Setting $F/P = 2$ in the formula for the number of periods to double, we get

$$N = \frac{\ln(F/P)}{\ln(1+i)} = \frac{\ln 2}{\ln(1+i)}.$$

Two approximations turn this into a rule that can be done in your head.

First, the numerator. The exact value is $\ln 2 = 0.6931$.

Second, the denominator. For a small rate, $\ln(1+i) \approx i$ (this is the first term of the series $\ln(1+i) = i - \frac{i^2}{2} + \cdots$). Because this drops the negative $i^2/2$ term, $\ln(1+i)$ is a little smaller than i, so replacing it by i alone makes N slightly too small. To compensate, we use a numerator slightly larger than 0.6931. The value 0.72 is convenient because 72 has many whole-number divisors $(2, 3, 4, 6, 8, 9, 12)$, which makes the mental arithmetic easy:

$$N = \frac{\ln 2}{\ln(1+i)} \approx \frac{0.72}{i} = \frac{72}{100\,i}.$$

Writing the rate as a whole-number percentage, $i\% = 100\,i$, this is simply

$$N \approx \frac{72}{i\%}.$$

For example, at $i = 8\%$ the rule gives $72/8 = 9$ years, while the exact value is $\ln 2/\ln 1.08 = 9.01$ years. The rounding up from 0.6931 to 0.72 and the approximation $\ln(1+i) \approx i$ pull in opposite directions and nearly cancel for rates of about 4% to 15%, which is why the Rule of 72 is accurate over that range. Figure 2.10 shows the two curves side by side: the Rule of 72 estimate hugs the exact doubling time across the highlighted band and only drifts noticeably outside it.

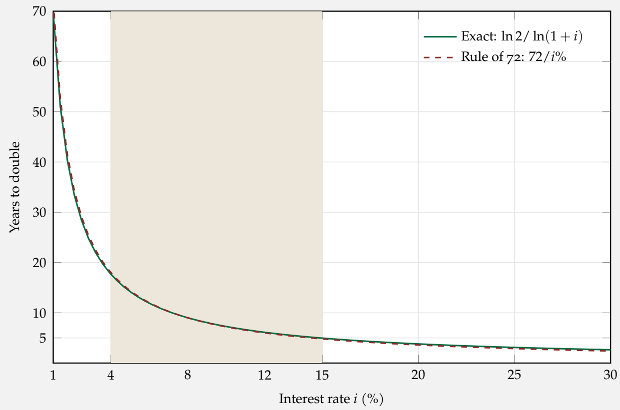

Figure 2.10. Exact doubling time vs. the Rule of 72. The two curves nearly coincide across the gold band ($i = 4\%$ to 15%); outside the band, the rule increasingly underestimates at low rates and overestimates at high rates.

2.10 *Uniform Series: Future Worth and Sinking Fund*

Lecture 5. This section is covered by Lecture 5.

Uniform series. An equal amount A paid at the end of every period, from period 1 to period N.

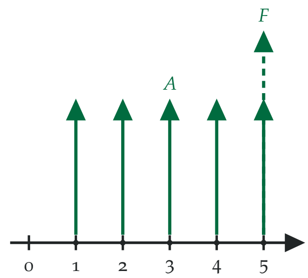

Figure 2.11. A uniform series. The first payment is at the end of period 1; F is measured at period N, at the same time as the last payment.

A *uniform series* is an equal amount A paid at the end of every period from period 1 to period N (Figure 2.11). Two conventions must be set before any formula is used: the first payment occurs at the *end* of period 1, not at time zero; and the future

In practice. A level series is also an engineering claim: it assumes level performance. A flotation cell's recovery drifts between relines and a truck's availability falls with age, so treat a constant A as an approximation to be justified, not as a default.

$(F/A, i, N)$. Read as "find F given A." It converts a whole series into its future amount.

amount F is measured at the same time as the last payment, at period N. Almost every error with series comes from placing one of these two points incorrectly.

2.10.1 *Equal-payment compound amount factor* $(F/A, i, N)$

$$F = A\left[\frac{(1+i)^N - 1}{i}\right] = A\,(F/A, i, N). \tag{2.5}$$

Insight. Where (F/A) comes from. Move each deposit forward to period N and add the results. The deposit at period 1 grows for $N-1$ periods, the deposit at period 2 grows for $N-2$ periods, and the last deposit does not grow at all:

$$F = A(1+i)^{N-1} + A(1+i)^{N-2} + \cdots + A(1+i) + A.$$

This is a geometric series with ratio $(1+i)$. Multiply the whole equation by $(1+i)$, then subtract the original equation from it. Every middle term cancels, which leaves $iF = A[(1+i)^N - 1]$. Dividing by i gives the factor. The single i in the denominator is the term that remains after the subtraction.

Sinking fund. A series of equal deposits set aside so that a known future expense is fully funded when it occurs. (A/F) gives the size of the deposit.

2.10.2 *Sinking-fund factor* $(A/F, i, N)$

This is the reverse of the previous case: given a future target F, find the equal deposit A needed to reach it. Because the sinking-fund factor is the reciprocal of (F/A), no new derivation is needed:

$$A = F\left[\frac{i}{(1+i)^N - 1}\right] = F\,(A/F, i, N). \tag{2.6}$$

It is on the same table row as (F/A), and the two factors multiply to one.

Example 2.11 — Funding an overhaul with a sinking fund

A haul-truck fleet will need a 100,000 SAR overhaul in 8 years. The maintenance account earns 7%. What equal end-of-year deposit fully funds the overhaul?

Solution.

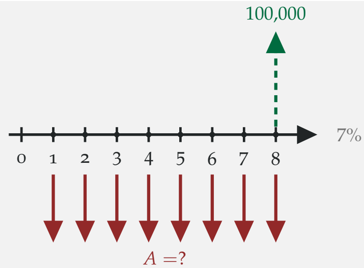

Figure 2.12. Eight equal deposits at years 1–8 reach the target at year 8.

Step 1: factor notation. $A = F\,(A/F, 7\%, 8)$.

N	(F/A)	(A/F)
6	7.1533	0.13980
7	8.6540	0.11555
8	10.2598	**0.09747**
9	11.9780	0.08349

Table 2.3. Excerpt from the $i = 7\%$ interest factor table used for Example 2.11.

Step 2: read the table. On the 7% uniform-series page, the row $N = 8$ across the (A/F) column (Table 2.3) gives 0.09747.

Step 3: substitute.

$$A = 100{,}000\,(0.09747) = \boxed{9{,}747 \text{ SAR per year}}.$$

The eight deposits total only 77,976 SAR of the company's own money; compound interest provides the remaining 22,024 SAR.

Example 2.12 — Why starting early wins

Two engineers both plan to retire in 40 years and both invest at 8% compounded annually. **Early Eve** deposits 6,000 SAR at the end of each of years 1–10, then stops contributing (but leaves the fund to grow). **Late Lina** contributes nothing for the first 30 years, then deposits 6,000 SAR at the end of each of years 31–40. Each contributes the same total cash, $10 \times 6{,}000 = 60{,}000$ SAR. Find each fund's balance at year 40 and explain the gap.

Solution.

Eve's fund.

The first ten deposits accumulate as a series, then the balance compounds forward for 30 more years:

$$F^{\text{Eve}} = 6{,}000\,(F/A,\,8\%,\,10)\,(F/P,\,8\%,\,30)$$
$$= 6{,}000\,(14.4866)(10.0627) = \boxed{874{,}610\text{ SAR}}.$$

Lina's fund.

The ten late deposits form a series whose target date is the last deposit (year 40), so:

$$F^{\text{Lina}} = 6{,}000\,(F/A,\,8\%,\,10) = 6{,}000\,(14.4866)$$
$$= \boxed{86{,}920\text{ SAR}}.$$

Eve ends with about $874{,}610/86{,}920 \approx 10\times$ Lina's balance, despite making the *same* total contribution. The difference is *when* the money was put to work: Eve's deposits compounded for 30 extra years, and exponential growth in those extra years dwarfs the late deposits. This is the most important practical consequence of $F = P(1+i)^N$.

Example 2.13— Mixing a lump sum with a uniform series

A plant places 30,000 SAR in an account today and adds 10,000 SAR at the end of every year for eight years. The account earns 8%. Find the balance just after the eighth deposit.

Solution.

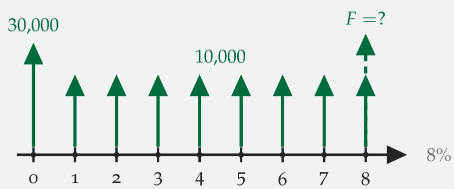

Figure 2.13. An opening deposit plus an eight-year series, both valued at year 8.

Move the lump sum forward 8 years with (F/P), and move the series to its last-deposit date with (F/A), then add:

$$\begin{aligned} F &= 30{,}000\,(F/P,\,8\%,\,8) + 10{,}000\,(F/A,\,8\%,\,8) \\ &= 30{,}000\,(1.8509) + 10{,}000\,(10.6366) \\ &= 55{,}527 + 106{,}366 = \boxed{161{,}893 \text{ SAR}}. \end{aligned}$$

Treating the two parts separately and adding them is the basic recipe for every composite cash flow in the chapter.

Example 2.14 — Sinking fund for a known target

What equal end-of-year deposit at 8%, for 10 years, builds a 150,000 SAR target fund?

Solution. The future target is known, so use the sinking-fund factor:

$$A = 150{,}000\,(A/F,\,8\%,\,10) = 150{,}000\,(0.06903) = \boxed{10{,}354 \text{ SAR per year}}.$$

The ten deposits total only 103,540 SAR; the rest of the target, $150{,}000 - 103{,}540 = 46{,}460$ SAR, is compound interest accrued on the earlier deposits.

Lecture 6. This section is covered by Lecture 6.

2.11 *Uniform Series: Present Worth and Capital Recovery*

2.11.1 *Capital-recovery factor $(A/P, i, N)$*

A loan of P is received today and repaid in N equal end-of-period installments:

$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right] = P\,(A/P, i, N). \tag{2.7}$$

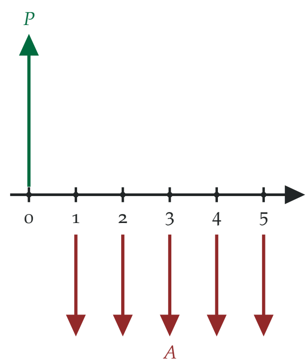

Figure 2.14. A loan P received now, repaid by N equal installments.

$(A/P, i, N)$**.** Read as "find A given P." It converts a present amount into equal installments. It is also the annual cost of owning an asset bought for P.

Insight. Why a loan payment is larger than the interest alone. Separate the numerator as $i(1+i)^N = i[(1+i)^N - 1] + i$, then divide. The factor becomes

$$(A/P, i, N) = i + \frac{i}{(1+i)^N - 1} = i + (A/F, i, N).$$

So every equal loan payment is the interest on the balance, i, plus a sinking-fund deposit, (A/F), that repays the principal by the end of the term. Early in the loan the balance is large, so the interest part is larger. Later the balance is small, so the principal part is larger. The payment stays equal only because the division between the two parts changes each period.

In practice. Venture debt is priced with this factor. A lender advancing P against 36 equal monthly payments has quoted $A = P\,(A/P, i, 36)$; the founder's job is to recover the implied i from the quoted payment, not to judge the payment's size alone.

2.11.2 *Present-worth factor* $(P/A, i, N)$

$(P/A, i, N)$**.** Read as "find P given A." It gives the present value of a whole series.

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] = A\,(P/A, i, N). \tag{2.8}$$

This factor and (A/P) are exact reciprocals on the same table row.

Insight. Where (P/A) comes from, and why its result is placed one period before the first payment. Move each payment back to time zero and add:

$$P = A(1+i)^{-1} + A(1+i)^{-2} + \cdots + A(1+i)^{-N}.$$

This is a geometric series in which the first term is moved back exactly *one* period. The sum is therefore measured at the point that is one period before the first payment, because that is the reference point from which the first term is moved back by a single period. For an ordinary series whose first payment is at period 1, "one period before the first payment" is period 0, which is why (P/A) normally gives a value at time zero. When the first payment is delayed, (P/A) still gives a value one period before that delayed payment, not at time zero. This fact is the complete explanation of the deferred-annuity rule below.

Example 2.15— Reading (P/A) from the table

A maintenance contract pays 9,000 SAR per year for 10 years. Money is worth 8%. Find the present worth.

Solution.

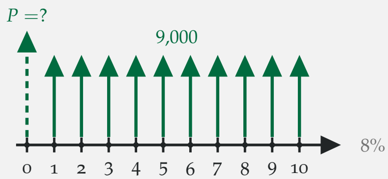

Figure 2.15. Ten equal payments; the present value is at year 0.

Step 1: factor notation. $P = A\,(P/A, 8\%, 10)$.

N	(A/P)	(P/A)
8	0.17401	5.7466
9	0.16008	6.2469
10	0.14903	**6.7101**
11	0.14008	7.1390

Table 2.4. Excerpt from the $i = 8\%$ interest factor table used for Example 2.15.

Step 2: read the table. On the 8% page, row $N = 10$, the (P/A) column (Table 2.4) gives 6.7101.

Step 3: substitute.

$$P = 9{,}000\,(6.7101) = \boxed{60{,}391 \text{ SAR}}.$$

Example 2.16 — Financing a conveyor with equal installments

A plant borrows 50,000 SAR at 8% to install a conveyor and will repay the loan in five equal end-of-year installments. Find the annual payment and the total interest paid over the loan.

Solution. A present loan repaid by equal installments uses the capital-recovery factor.

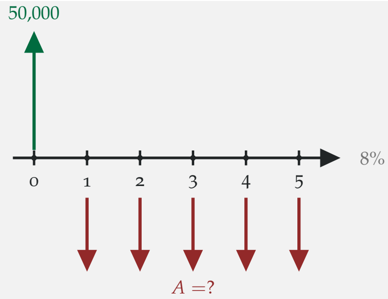

Figure 2.16. Loan received now, repaid by five equal end-of-year payments.

$$A = 50{,}000\,(A/P,\,8\%,\,5) = 50{,}000\,(0.25046) = \boxed{12{,}523\text{ SAR per year}}.$$

Total paid over the loan is $5 \times 12{,}523 = 62{,}615$ SAR, so the interest paid is $62{,}615 - 50{,}000 = 12{,}615$ SAR. The plant exchanges the use of the 50,000 SAR today for that fixed interest cost.

Example 2.17 — Capital recovery on a small loan

A team borrows 20,000 SAR at 7% and repays it in five equal end-of-year installments. Find the annual payment.

Solution.

$$A = 20{,}000\,(A/P,\,7\%,\,5) = 20{,}000\,(0.24389) = \boxed{4{,}877.80\text{ SAR per year}}.$$

Five payments of 4,877.80 total 24,389 SAR; the difference, 24,389 - 20,000 = 4,389 SAR, is the interest paid for the use of the money over five years.

Perpetuity. A series that pays A every period forever. Its present value is $P = A/i$.

2.11.3 *Perpetuity: a series that continues forever*

When the payments continue forever, the present worth does not increase without limit; it approaches a finite value:

$$P = \frac{A}{i}. \tag{2.9}$$

Insight. Why an unending series has a finite value. The present worth is the infinite sum

$$P = A(1+i)^{-1} + A(1+i)^{-2} + A(1+i)^{-3} + \ldots,$$

a geometric series with first term $a = A(1+i)^{-1}$ and ratio $r = (1+i)^{-1}$. Any positive interest rate makes $r = 1/(1+i) < 1$, so each term is a smaller fraction than the one before it, and the series converges. Its sum is $a/(1-r)$, which simplifies to A/i. The interest rate is the reason the ratio is less than one, so the rate is also the reason the infinite sum is finite. If the rate were zero, the ratio would equal one, the terms would not decrease, and the value would be infinite. This is the mathematical meaning of money that has no time value.

2.11.4 *Deferred annuity: a series that begins later*

When a series does not begin until after a delay, use the two-step rule. If the first of N payments occurs at period $d+1$, then (P/A) gives a value at period d, and that single amount is then moved back to time zero:

$$P_0 = A\,(P/A, i, N)\,(P/F, i, d). \tag{2.10}$$

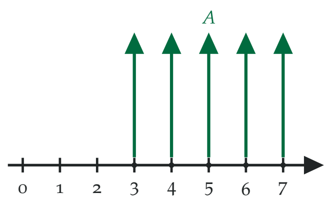

Figure 2.17. A deferred annuity. (P/A) gives a value at period 2, one period before the first payment at period 3; (P/F) then moves it to time zero.

Deferred annuity. A uniform series whose first payment is delayed. Value it in two steps: use (P/A) to reach one period before the first payment, then use (P/F) to reach time zero.

In practice. The delay d is usually a construction and commissioning duration. Underestimating it shifts the whole series to the right, and every period of slip discounts every payment by one more factor of $(1+i)^{-1}$.

Example 2.18 — Valuing a delayed lease

A lease pays 2,000 SAR per year at the ends of years 3 through 7, at 8% (Figure 2.17). There are five payments, and the first is at year 3.

Solution. We work in two steps, and we draw the cash flow after each step so the effect of each factor is visible.

Step 1: use (P/A) on the five payments.

The factor gives a value one period before the first payment, at year 2.

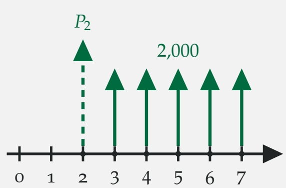

Figure 2.18. After Step 1, the five payments become a single amount P_2 at year 2.

N	(P/A)	(P/F)
2	1.7833	**0.8573**
3	2.5771	0.7938
4	3.3121	0.7350
5	**3.9927**	0.6806

Table 2.5. Excerpt from the $i = 8\%$ interest factor table used for Example 2.18.

$$P_2 = 2{,}000\,(P/A, 8\%, 5) = 2{,}000\,(3.9927) = 7{,}985 \text{ SAR}.$$

Step 2: use (P/F) to move P_2 to time zero.
The single amount at year 2 is moved back two years.

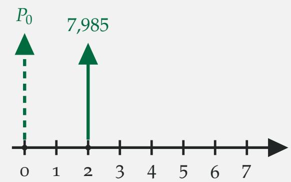

Figure 2.19. After Step 2, P_2 becomes the present value P_0 at year 0.

$$P_0 = 7{,}985\,(P/F, 8\%, 2) = 7{,}985\,(0.8573) = \boxed{6{,}846 \text{ SAR}}.$$

Lecture 7. This section is covered by Lecture 7.

2.12 *Gradient Series*

Real cash flows are rarely level. Two structured patterns cover most cases: a flow that grows by a constant *amount* each period (a linear gradient), and a flow that grows by a constant *percentage* each period (a geometric gradient).

2.12.1 *The strict linear gradient* $(P/G, i, N)$

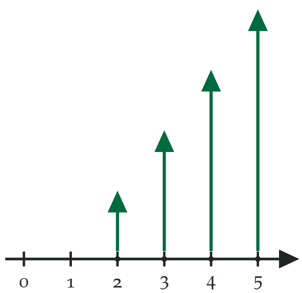

Figure 2.20. A strict linear gradient. There is no arrow at period 1; the first nonzero flow, G, is at period 2.

Strict linear gradient. A series that is zero in period 1 and rises by a constant G each period: $0, G, 2G, \ldots, (N-1)G$.

A strict linear gradient starts at zero and increases by a constant amount G each period (Figure 2.20):

$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right] = G\,(P/G, i, N). \tag{2.11}$$

Insight. Why the first flow is zero. The gradient factor represents only the *growth* in a series, not its starting level. The amount in period n is $(n-1)G$: no growth has occurred by period 1, one step has been added by period 2, two steps by period 3, and so on. The first term of the present-worth sum is therefore $0 \cdot (1+i)^{-1}$, which adds nothing. This is not a modeling choice. A step is the *difference* between two consecutive periods, and there is no earlier period for the first one to differ from. This is why the table always lists $(P/G, i, 1) = 0$.

2.12.2 *Realistic growth: uniform plus gradient*

A series that increases by a fixed amount but starts above zero is the sum of two parts: a level uniform series for the base amount that is present in every period, valued with (P/A), and a strict gradient for the increase, valued with (P/G).

Example 2.19 — Separating a growing revenue

Revenue is 120,000 SAR in year 1 and increases by 18,000 SAR each year through year 5, at 10%. Find the present worth.

Solution. We separate the series into its two parts and draw each one.

Part 1: the level base of 120,000 each year.

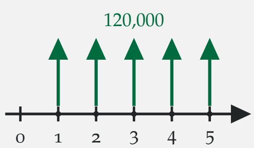

Figure 2.21. The base amount, present in every year.

N	(P/A)	(P/G)
3	2.4869	2.3291
4	3.1699	4.3781
5	**3.7908**	**6.8618**
6	4.3553	9.6842

Table 2.6. Excerpt from the $i = 10\%$ interest factor table used for Example 2.19.

$$120{,}000\,(P/A, 10\%, 5) = 120{,}000\,(3.7908) = 454{,}896 \text{ SAR}.$$

Part 2: the gradient of 18,000, which is zero in year 1.

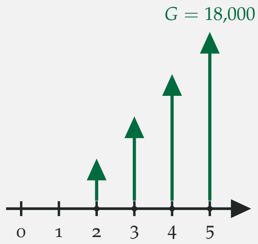

Figure 2.22. The increase, 0 in year 1, then 18,000, 36,000, and so on.

$$18{,}000\,(P/G, 10\%, 5) = 18{,}000\,(6.8618) = 123{,}512 \text{ SAR}.$$

Add the two parts.

$$P = 454{,}896 + 123{,}512 = \boxed{578{,}408 \text{ SAR}}.$$

Example 2.20 — Rising maintenance: pure linear gradient

A piece of equipment has zero maintenance cost in its first year, then 1,000 SAR more in each successive year through year 5. With $i = 10\%$, find the present worth of the five-year maintenance schedule.

Solution. This is a strict linear gradient with $G = 1{,}000$: the costs are 0, 1,000, 2,000, 3,000, 4,000 in years 1–5.

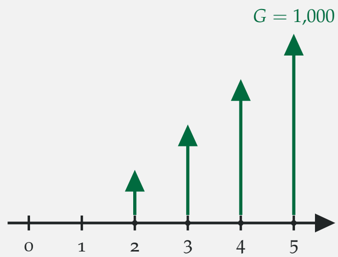

Figure 2.23. Strict linear gradient: zero in year 1, then growing by G per year.

$$P = 1{,}000\,(P/G,\ 10\%,\ 5) = 1{,}000\,(6.8618) = \boxed{6{,}862 \text{ SAR}}.$$

The first nonzero cost is at year 2; the present-worth factor (P/G) accounts for the year-1 zero automatically (this is why the table lists $(P/G, i, 1) = 0$).

Example 2.21 — Decomposed growth, try-by-hand

A new product is forecast to bring in 60,000 SAR in year 1 and grow by 4,000 SAR each year through year 5. With $i = 9\%$, find the present worth of the revenue stream.

Solution. Separate the series into a level base of 60,000 (uniform) and an increase of 4,000 starting in year 2 (strict gradient):

$$\begin{aligned} P &= 60{,}000\,(P/A,\,9\%,\,5) + 4{,}000\,(P/G,\,9\%,\,5) \\ &= 60{,}000\,(3.8897) + 4{,}000\,(7.1110) \\ &= 233{,}382 + 28{,}444 = \boxed{261{,}826 \text{ SAR}}. \end{aligned}$$

The base contributes about 89% of the present worth; the gradient's share grows when either the period N or the per-year step G becomes larger.

Geometric gradient. A series in which each period's cash flow is the previous one multiplied by $(1+g)$. The growth is by a constant percentage.

2.12.3 *The geometric gradient*

When growth is by a constant percentage, the cash flow in year n is $A_1(1+g)^{n-1}$, and

$$P = A_1\left[\frac{1-(1+g)^N(1+i)^{-N}}{i-g}\right] \quad (i \neq g), \qquad P = \frac{A_1 N}{1+i} \quad (i = g). \tag{2.12}$$

Insight. Where the $i = g$ case comes from. Moving payment n back to time zero gives $A_1(1+g)^{n-1}(1+i)^{-n}$, and factoring leaves a geometric series with ratio $r = (1+g)/(1+i)$. When $g \neq i$, the ratio is not one and the usual sum applies. But when $g = i$, the ratio is exactly one, and the usual formula would divide by $1 - r = 0$, which is undefined. In that case the sum is simply N copies of the same value, which gives $P = A_1N/(1+i)$. The financial meaning is that growth and discounting cancel each other in every period, so every payment has the same present value, and the total is that value repeated N times.

Example 2.22 — Geometric gradient with declining cash flows

An operating cost is 25,000 SAR in year 1 and *decreases* by 5% each year through year 5. With $i = 10\%$, find the present worth of the five years of costs.

Solution. The cash flow is a geometric series with $A_1 = 25{,}000$, $g = -0.05$ (negative because the cost falls), $i = 0.10$, $N = 5$. The formula handles $g < 0$ exactly the same as $g > 0$:

$$\begin{aligned} P &= A_1 \frac{1 - (1+g)^N (1+i)^{-N}}{i - g} \\ &= 25{,}000 \, \frac{1 - (0.95)^5 (1.10)^{-5}}{0.10 - (-0.05)} \\ &= 25{,}000 \, \frac{1 - (0.7738)(0.6209)}{0.15} = 25{,}000 \, \frac{1 - 0.4805}{0.15} \\ &= 25{,}000 \, (3.4634) = \boxed{86{,}585 \text{ SAR}}. \end{aligned}$$

Because $g < 0$, the denominator $i - g = 0.15$ is *larger* than for a flat series, and the present worth comes out smaller than the $(P/A, 10\%, 5) \times 25{,}000 = 94{,}769$ SAR that a constant-25,000 series would give. The falling cost saves the firm about $94{,}769 - 86{,}585 = 8{,}184$ SAR in present-worth terms.

2.13 *The Strategy for Composite Cash Flows*

Any cash flow stream, however complex, is a sum of the four basic types developed in this chapter: single flows, handled by (P/F) or (F/P); uniform series, handled by (P/A) or (F/A), where (P/A) gives a value one period before the first payment; linear gradients, handled by (P/G), where the first flow is zero; and geometric series, handled by the A_1 formula, with attention to the $i = g$ case.

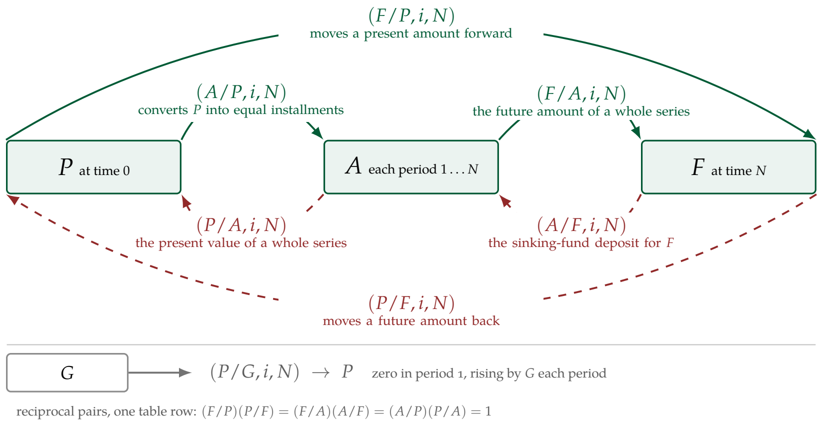

Figure 2.24. The whole factor system on one map. Every factor is an arrow between two of the three quantities P, A, and F: an arrow running to the right carries value forward in time and an arrow running to the left carries it back, and the two arrows on any pair are reciprocals read off the same table row.

Figure 2.24 puts every factor of this chapter on one map, and it is the figure to come back to whenever you are unsure which factor a problem needs: each pair of quantities carries two arcs, one in each direction, and those two arcs are reciprocals taken from the same table row.

> **Insight.** The method is always the same: *draw* the diagram, *separate* it into basic parts, *value* each part at a common date, then *add*. A full solution should show all of these steps, and this is also what earns partial credit on an exam.

2.14 *Case Study from the Literature*

Published techno-economic studies are built from exactly the pieces developed in this chapter: a first cost at year 0, a uniform operating cost, a single replacement in a later year, and a revenue stream that changes at a constant percentage. The case below takes the published inputs of a real study and re-derives its headline result with the four basic factors.

Reference. Imam, Al-Turki, and Sreerama Kumar (2020), *Sustainability*, 12(1), 262. Open access; the full cost and performance data are in the paper's Section 2.

Example 2.23 — Case study: a rooftop solar system in Jeddah

Imam, Al-Turki, and Sreerama Kumar (2020) assessed a 12.25-kW rooftop photovoltaic system for a residential building in Jeddah, using renewable-energy simulation software. The study is in US dollars, so this example keeps the paper's currency. Their published inputs are: a total capital cost of $10,529.46 at year 0; an operation and maintenance cost of $16 per kW per year, which is $16 \times 12.25 = \$196$ per year for the 25-year life; an inverter replacement of $1,346.79 at year 15; a first-year output of 23,589 kWh that declines by 0.5% per year as the modules degrade; an electricity tariff of $0.048 per kWh for monthly consumption below 6,000 kWh (the building's usage tier); and an interest rate of 2.5%. The study reported a net present value of $4,378 and a discounted payback of 14.6 years. Re-derive the present worth with this chapter's factors and compare it with the published result.

Solution.

Step 1: draw the diagram and separate it.

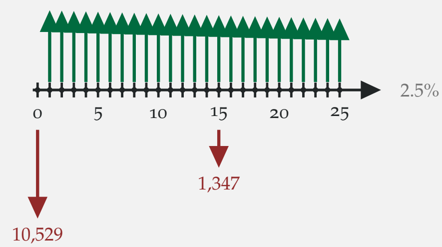

Figure 2.25. The published cash flows: capital at year 0, energy savings declining 0.5% per year, and the inverter replacement at year 15. The $196 O&M series is netted against the savings in the calculation.

The stream has four basic parts: a single outflow at year 0; a geometric series of energy savings (first amount A_1, growth rate $g = -0.005$); a uniform O&M series of $196 per year; and a single outflow at year 15.

Step 2: value the energy savings (geometric gradient).

The first-year saving is $A_1 = 23{,}589 \times 0.048 = \$1{,}132.27$. With $i = 0.025$, $g = -0.005$, and $N = 25$, the denominator is $i - g = 0.030$:

$$P_{\text{energy}} = 1{,}132.27 \left[\frac{1 - (0.995)^{25}(1.025)^{-25}}{0.030} \right] = 1{,}132.27\,(17.4713) = 19{,}782\,\text{USD}.$$

Step 3: value the O&M series and the inverter replacement.

$$P_{\text{O\&M}} = 196\,(P/A,\, 2.5\%,\, 25) = 196\,(18.4244) = 3{,}611 \text{ USD},$$
$$P_{\text{inverter}} = 1{,}346.79\,(P/F,\, 2.5\%,\, 15) = 1{,}346.79\,(0.6905) = 930 \text{ USD}.$$

Step 4: add the parts at year 0.

$$P = 19{,}782 - 3{,}611 - 930 - 10{,}529.46 = \boxed{+4{,}712 \text{ USD}}.$$

The four factors reproduce the paper's simulated \$4,378 to within about 8%. The remaining gap is expected: the software also modeled a five-year loan, an inflation rate of -1.8%, and month-by-month billing, none of which the four basic factors carry. A year-by-year check of the same numbers shows the discounted balance first turning positive at year 14, dipping negative again when the year-15 inverter replacement is paid, and staying positive from year 16 onward, which is consistent with the paper's 14.6-year discounted payback.

The researchers concluded that the system is economically feasible: the net present value is positive and the levelized cost of its energy (\$0.0382 per kWh) is below the utility tariff. They also noted that the long payback discourages household investors, and that a time-of-use billing scheme would raise the net present value to \$9,281 and shorten the discounted payback to about 10.5 years (Imam et al., 2020).

Insight. A published feasibility study is the composite strategy of section 2.13 at work. Once the paper's inputs are placed on one timeline, its headline number follows from three factors and one formula; the simulation software adds refinement, not a different method. This is why an engineer who knows the factors can audit a study without owning the software that produced it.

2.15 Chapter Summary

- Money has a **time value**, which comes from earning power and inflation and is measured by the interest rate.
- Any interest transaction uses five elements: P, i, N, A, F.
- The **cash flow diagram** shows money over time: inflows up, outflows down, the end-of-period convention, and one point of view.
- **Simple** interest grows linearly, $F = P(1 + iN)$; **compound** interest grows exponentially, $F = P(1 + i)^N$. The course uses compound interest.
- **Equivalence** means equal value at a common date for a given rate, which means the holder is indifferent.
- **Single-payment factors**: (F/P) moves forward, (P/F) moves back; they are reciprocals.
- **Uniform-series factors**: (F/A) and (A/F) for future amount and sinking funds; (A/P) and (P/A) for installments and present worth; a perpetuity is A/i.
- **Deferred annuity**: use (P/A) to reach one period before the first payment, then (P/F) to reach time zero.
- **Gradients**: (P/G) for a fixed-amount increase (first flow zero, usually separated into uniform plus gradient); the A_1 formula for a fixed-percentage increase.
- For any composite stream: draw, separate, value at a common date, and add.

2.16 Exercises

The problems run in factor order: single flows first, then mixed timelines, uniform series, deferred and gradient series. The later sets combine several factors on one timeline, including break-even rates and a full project cash flow schedule.

Single flows

Exercise 2.1. A robotics research fund of 100,000 SAR is invested at 7% compounded annually. (a) How much is in the account after 8 years? (b) How much of that amount is interest earned on earlier interest, that is, the amount above what simple interest would give?

Exercise 2.2. An EV-charging operator will invest 20,000 SAR for 6 years. Bank A pays 9% simple interest; Bank B pays 8% compounded annually. Which account has more money at the end?

Exercise 2.3. A training endowment deposits 50,000 SAR at 8% compounded annually. How many years until it reaches 200,000 SAR?

Exercise 2.4. A scholarship fund grows from 25,000 SAR to 40,000 SAR in 6 years. What annual rate did it earn?

Exercise 2.5. A coffee-shop replacement fund earns 10% compounded annually. How many years until it grows to three times its value?

Mixing single flows on one timeline

Exercise 2.6. A minerals fund deposits 100,000 SAR now and 200,000 SAR two years from now, at 10%. Find the balance at the end of year 10.

Exercise 2.7. A plant keeps an automation reserve that earns 7% compounded annually. It deposits 10,000 SAR now, deposits 5,000 SAR at the end of year 2, withdraws 4,000 SAR at the end of year 4, and deposits 6,000 SAR at the end of year 5. (a) Find the balance at the end of year 8. (b) The plant then withdraws the whole balance in four equal amounts at the ends of years 9–12. Find the amount of each withdrawal.

Uniform series

Exercise 2.8. A jewelry store saves 5,000 SAR at the end of each year for 5 years at 6%. What is the balance just after the fifth deposit?

Exercise 2.9. A flotation-cell replacement will cost 250,000 SAR in 5 years; the reserve account earns 8%. What equal end-of-year deposit fully funds the replacement?

Exercise 2.10. A premium hotel expects level annual housekeeping savings of 40,000 SAR for 8 years. At 10%, what is the present worth of these savings at year 0?

Exercise 2.11. A phosphate plant borrows 60,000 SAR at 9% compounded annually to install a conveyor. **Plan 1** repays the loan in five equal year-end payments. **Plan 2** repays the loan in a single payment at the end of year 5. (a) Find the Plan 1 payment. (b) Find the Plan 2 payment. (c) Report the total paid and the interest paid under each plan. (d) Explain why the lender is indifferent between the two plans.

Exercise 2.12. A research foundation will fund a robotics lab either with 200,000 SAR now (Option A) or with 28,000 SAR at the end of each year for 12 years (Option B). (a) At 6%, find the present worth of Option B and state which option is worth more today. (b) Find the interest rate at which the two options are equivalent, and explain what it means.

The break-even (indifference) rate

Exercise 2.13. A kids club can deposit a single 19,000 SAR now, or set aside 6,000 SAR at the end of each year for 4 years. At what interest rate do the two plans have the same value at year 4?

Exercise 2.14. Two payment plans for new courts are equal in value at 10%: Plan P is 50,000 SAR now; Plan Q is 13,190 SAR per year for 5 years. If the market rate is actually 12%, which plan has the lower present worth?

Choosing between two future dates

Exercise 2.15. A padel club will receive a 50,000 SAR sponsorship at year 8, or the same amount at year 10. Which is more valuable today, and why? Money is worth 8%.

Uneven savings: present worth at year zero

Exercise 2.16. A robot upgrade is expected to save 40,000 SAR in year 1, 15,000 SAR in year 2, and 60,000 SAR in year 3. At 8%, what is the most the team should pay for the upgrade today?

Gradient series

Exercise 2.17. A lithium refinery projects its first five years of revenue in two ways, at 10%. **Projection G (geometric):** 120,000 SAR in year 1, increasing 15% per year through year 5. **Projection L (linear):** 120,000 SAR in year 1, increasing by 18,000 SAR each year through year 5. (a) Find the present worth of Projection G. (b) Find the present worth of Projection L. (c) State which is worth more today, and by how much.

Rate of return against a hurdle

Exercise 2.18. A smart-vehicle-fleet team can invest 90,000 SAR to expand its operation, and the expanded business is expected to be worth 250,000 SAR after 6 years. The team's minimum acceptable rate of return (MARR) is 15%. Is the investment acceptable, on a rate-of-return basis?

The project cash flow schedule (in-class activity)

Exercise 2.19. A project has the cash flow schedule below, at $i = 8\%$ compounded annually. *Outflows:* year 0, $-90{,}000$; year 1, $-120{,}000$; year 2, $-25{,}000$; year 4, $-40{,}000$; year 7, $-200{,}000$; year 12, $-150{,}000$. *Inflows:* year 3, $+60{,}000$; year 6, $+110{,}000$; year 9, $+80{,}000$; year 11, $+130{,}000$; year 15, $+170{,}000$; year 20, $+30{,}000$. (a) Find the total present worth at year 0. (b) Move that total forward to year 10 under compound interest and under simple interest, and report the difference.

Exercise 2.20. An investor has 500,000 SAR. Strategy A earns 12% *simple* interest; Strategy B earns 10% *compounded* annually. How long does each strategy take to double the money, and which would you choose?

Exam-style problems

The following problems are written in the format of Major Exam 1: a scenario, lettered parts with point values, and full work expected. Draw the cash flow diagram, write the factor notation before substituting numbers, box each final answer with its units, and end with a decision sentence.

Exercise 2.21. A liner-supply contract at Najd Copper. Najd Copper signs a crusher-liner supply contract with the following cash flows: 120,000 SAR *received* at the end of year 0 (mobilization), 90,000 SAR *paid* at the end of year 2 (liner purchase), 150,000 SAR *received* at the end of year 4 (milestone payment), and 200,000 SAR *received* at the end of year 6 (completion payment). The firm's money is worth 8% compounded annually. (a) (*4 pts*) Draw a labeled cash flow diagram of the four contract flows. (b) (*8 pts*) Find the single amount at the end of year 6 that is equivalent to all four flows. (c) (*8 pts*) A trading house offers to buy the whole contract for 320,000 SAR paid today. Find the contract's equivalent value today and state whether the firm should sell.

Exercise 2.22. Funding a flotation-cell rebuild at Wadi Phosphate. A flotation-cell rebuild will cost 260,000 SAR at the end of year 7. The reserve account earns 7% compounded annually. (a) (*6 pts*) Plan 1 makes equal deposits at the ends of years 1–7. Find the required deposit. (b) (*6 pts*) Under Plan 1, find the account balance just after the fourth deposit. (c) (*8 pts*) Budget pressure delays the start: Plan 2 makes equal deposits at the ends of years 3–7 only. Find the new deposit, compare the total of the firm's own money deposited under each plan, and state which plan to adopt.

Exercise 2.23. Haul-road maintenance at Wadi Lithium. Maintenance on a lithium-mine haul road is expected to cost 40,000 SAR in year 1 and to rise by 6,000 SAR each year through year 6 as the road ages. Money is worth 10%. (a) (*4 pts*) Draw the cash flow diagram, separating the series into its level base and its gradient. (b) (*10 pts*) Find the present worth of the six years of maintenance. (c) (*6 pts*) A contractor offers to take over all six years of maintenance for a single payment of 225,000 SAR today. State whether the mine should sign.

Assessing outside recommendations

The two problems below present finished solutions produced by others. In each, the arithmetic is internally consistent; the flaw is in the method, so only an auditor who understands the factors can catch it. For each problem, (a) decide whether to accept the recommendation, (b) identify each error or omission precisely, and (c) produce the corrected number.

Exercise 2.24. An AI solution for a water-truck lease. Wadi Lithium can lease a water truck for five annual payments of 18,000 SAR, with the *first payment due at signing* (year 0) and the remaining four at the ends of years 1–4. Buying the same truck outright costs 72,000 SAR today. Money is worth 10%. An AI assistant produced the solution below.

> *The lease is a uniform series of five payments of 18,000 SAR. Its present worth is* $P = 18{,}000\,(P/A, 10\%, 5) = 18{,}000\,(3.7908) = 68{,}234$ *SAR. Since* $68{,}234 < 72{,}000$, *the lease is cheaper than buying. Recommendation: lease.*

(a) (*4 pts*) Should the firm accept the recommendation? (b) (*8 pts*) Identify the error precisely. (c) (*8 pts*) Compute the correct present worth of the lease and state the correct recommendation.

Exercise 2.25. A dealer's add-on financing memo. A dealer offers to finance an 80,000 SAR ore-sample preparation unit for Najd Copper's laboratory and sends the memo below. A bank offers the alternative: a loan at 9% compounded annually, repaid in six equal end-of-year payments.

> *Our plan charges plain 9% interest for six years. Payment: principal portion* $80{,}000/6 = 13{,}333.33$; *interest portion* $0.09 \times 80{,}000 = 7{,}200$; *total* $20{,}533.33$ *SAR per year for six years. This is the same 9% per year that a bank's amortized loan charges, with simpler arithmetic.*

(a) (*4 pts*) Should the firm accept the dealer's claim? (b) (*8 pts*) Identify the flaw in the memo precisely. (c) (*8 pts*) Compute the correct payment on the bank's 9% amortized loan, and estimate the interest rate the dealer's plan actually charges.

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