ISE 307: Engineering Economic Analysis

Lecture 8: Nominal Rate, Effective Rate, and Mismatched Periods (Ch 3)

Term 253 (Summer)

Instructor: Mansur M. Arief
Industrial and Systems Engineering, KFUPM

Office: 22-219 — Email: mansur.arief@kfupm.edu.sa

Lecture 8

Nominal vs. Effective Rates, Mismatched Periods, and Comparing Financing Offers

Lecture 8 Objectives

1. Decode a quote: **nominal rate** r vs. **compounding frequency** M
2. Compute the **effective annual rate** i_a, and compare offers fairly
3. Handle **continuous** compounding ($i_a = e^r - 1$)
4. When periods **don't match**: effective rate **per payment period**
5. Apply it to loans: payment, **remaining balance**, interest vs. principal

Nominal vs. Effective

"9% compounded monthly" is not 9%

A quote of "r **compounded** M **times a year**" means r/M is charged *each period*:

$$\text{Rate per period: } i = \frac{r}{M} \qquad \text{Effective annual: } i_a = \left(1 + \frac{r}{M}\right)^M - 1$$

$r = 9\%$ **under different compounding frequencies:**

M	1 (annual)	2	4	12	52	365
i_a	9.000%	9.203%	9.308%	9.381%	9.409%	9.416%

The quoted 9% (**APR**) is *nominal* / monthly; you actually pay 9.38% (**effective APY**).

Continuous Compounding

The limit as $M \to \infty$

As compounding gets more and more frequent, the effective rate approaches a ceiling:

$$i_a = \lim_{M \to \infty} \left(1 + \frac{r}{M}\right)^M - 1 = e^r - 1$$

- e.g. 9% compounded **continuously**: $i_a = e^{0.09} - 1 = 9.417\%$
- This is the ceiling of the table above — daily (9.416%) is already almost there
- Compounding more often than continuously is impossible: $e^r - 1$ is the most frequency can do

Comparing Offers

Effective rates make different quotes comparable

A minerals supplier needs financing. Two quotes arrive:

- **Bank A:** 9.0% compounded **quarterly** $\Rightarrow$ $i_a = (1.0225)^4 - 1 = \mathbf{9.308\%}$
- **Bank B:** 8.9% compounded **monthly** $\Rightarrow$ $i_a = \left(1 + \frac{0.089}{12}\right)^{12} - 1 = \mathbf{9.272\%}$

Bank B is cheaper *despite* more frequent compounding — here the lower nominal rate wins. **Never compare nominal rates across different M; always convert to i_a first.**

Decision: *"Borrow from Bank B; its effective rate 9.27% is below Bank A's 9.31%."*

When Payments and Compounding Don't Match

Real life: you pay on one schedule, the bank compounds on another.

- Count three numbers: K = payments/yr, M = compoundings/yr
- $C = \dfrac{M}{K}$ = number of compoundings **inside one payment period**
- Examples: quarterly pay, monthly compounding $\Rightarrow K{=}4,\ M{=}12,\ C{=}3$

The factor tables need the rate **per payment period**. So we use the rate that matches *when the payments actually happen*.

Effective Rate per Payment Period

Align the rate with the payments, then use ordinary factors

Effective rate **per payment period**: $i = \left(1 + \dfrac{r}{M}\right)^{C} - 1$

$(C = M/K)$

Then use $(F/A, i, N)$, $(P/A, i, N)$, $(A/P, i, N)$, ... with $N =$ **number of payments**.

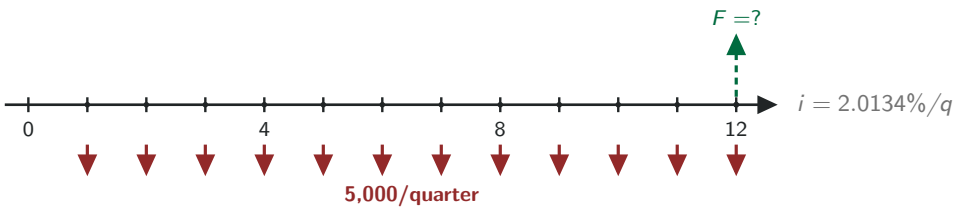
Example: pay **quarterly** (K=4), compounds **monthly** at $r = 8\%$ (M=12, C=3):

$$i = \left(1 + \tfrac{0.08}{12}\right)^{3} - 1 = (1.006667)^{3} - 1 = \boxed{2.0134\% \text{ per quarter}}$$

Check: more compounding inside the quarter ⇒ slightly more than the naive $8\%/4 = 2\%$. ✓

Example: A Quarterly Savings Plan

A maintenance fund saves **5,000 SAR per quarter** for 3 years; the account pays **8% compounded monthly**.



$$F = 5{,}000\,(F/A,\ 2.0134\%,\ 12) = 5{,}000\,(13.4221) = \boxed{\text{SAR } 67{,}110}$$

12 payments (not 36!) at the **quarterly** effective rate — N and i must always match.

Continuous Compounding, Discrete Payments

Effective rate per payment period under continuous compounding: $i = e^{r/K} - 1$

Example: weekly deposits ($K = 52$) at $r = 5\%$ compounded **continuously**:

$$i = e^{0.05/52} - 1 = \boxed{0.09619\% \text{ per week}}$$

Then any weekly-series factor applies, e.g. a 2-year plan uses $N = 104$ weekly periods.

Edge case: *if you deposit more often than the bank compounds, funds often earn nothing until the next compounding date — read the account rules first.*

Loans: Payment, Remaining Balance, Interest vs. Principal

Borrow P, repay with N equal payments at rate i **per payment period**:

- **Payment:** $A = P\,(A/P,\, i,\, N)$
- **Remaining balance after payment** t = present worth of the payments *still left*:
  $B_t = A\,(P/A,\, i,\, N - t)$
- **Split of the next payment:** interest $= i \cdot B_t$; principal $= A - i \cdot B_t$

The remaining balance is **always** the value today of the payments that haven't been made yet. That one idea answers every "balance after k payments" question.

Example: Financing a Roasting Line

Mismatched periods + amortization

Borrow **250,000 SAR**; **9% compounded monthly**; repay in **60 monthly payments**.

- Periods match (pay monthly, compound monthly): $i = 9\%/12 = 0.75\%$/month, $N = 60$
- **Payment:** $A = 250{,}000\,(A/P,\ 0.75\%,\ 60) = 250{,}000(0.020758) = \boxed{5{,}190\ \mathrm{SAR}}$
- **Balance after the 24th payment:**
  $B_{24} = 5{,}190\,(P/A,\ 0.75\%,\ 36) = \boxed{163{,}196\ \mathrm{SAR}}$
- **25th payment split:** interest $= 0.0075(163{,}196) = 1{,}224$; principal $= 5{,}190 - 1{,}224 = 3{,}966$

Reference: Effective Rates by Frequency

Compounding	$r = 6\%$	$r = 9\%$	$r = 12\%$
Annual (M=1)	6.000%	9.000%	12.000%
Quarterly (M=4)	6.136%	9.308%	12.551%
Monthly (M=12)	6.168%	9.381%	12.683%
Daily (M=365)	6.183%	9.416%	12.747%
Continuous	**6.184%**	**9.417%**	**12.750%**

How to read it:

- Each row a frequency; each column a rate
- The annual-to-continuous gap **widens with** r
- Continuous ($= e^r - 1$) is always the ceiling

Practice Problem 1

A credit line for shop equipment is quoted at **18% APR, compounded monthly**. What is the effective annual rate, and what does a 10,000 SAR balance cost over one year?

Apply the effective-rate formula ($r = 0.18$, $M = 12$):

$$i_a = \left(1 + \tfrac{0.18}{12}\right)^{12} - 1 = (1.015)^{12} - 1 = \boxed{19.56\%}$$

A 10,000 balance carried one year costs ≈**1,956 SAR**, not the 1,800 the headline 18% suggests.

Practice Problem 2

A KFUPM lab compares 3 equipment-loan offers. Rank from cheapest to expensive.

Offer	Quote (nominal rate)	i_a (effective rate)	rank
A	8.0% compounded monthly	8.300%	2
B	8.1% compounded quarterly	8.349%	3
C	8.2% compounded annually	8.200%	**1**

Order by i_a: **C (8.200%)** $<$ **A (8.300%)** $<$ **B (8.349%)**.
The highest *nominal* rate (C) is the cheapest after compounding.

Practice Problem 3

A 10,000 SAR loan is repaid with **5 equal annual payments** at **5% compounded weekly**. Find the annual payment.

Step 1 — effective annual rate (K=1, M=52, C=52):

$$i = \left(1 + \tfrac{0.05}{52}\right)^{52} - 1 = \boxed{5.125\% \text{ per year}}$$

Step 2 — payment with $(A/P,\ 5.125\%,\ 5)$:

$$A = 10{,}000\,(A/P,\ 5.125\%,\ 5) = 10{,}000\,(0.23177) = \boxed{\text{SAR } 2{,}318}$$

Summary

- A quote has **two parts**: nominal r and frequency M; per-period rate is r/M
- **Effective annual** $i_a = (1 + r/M)^M - 1$ is the true cost (APR vs. APY); compare offers **only** via i_a
- **Continuous** limit: $i_a = e^r - 1$ (the ceiling)
- Periods differ? **Effective rate per payment period**: $i = (1 + r/M)^C - 1$, $C = M/K$
- Then use chapter-2 factors with $N =$ **number of payments**
- Loans: $A = P(A/P)$; balance $B_t = A(P/A,\, i,\, N-t)$; interest $= i\, B_t$

Preparing for Major Exam I

What to be fluent in:

- **Ch. 1:** engineering economics, types of decisions, **principles** of engineering economics decision-making
- **Ch. 2:** single sums, series ($F/A, P/A, A/P$), gradients, equivalence on a **common date**
- **Ch. 3 (today):** nominal vs. effective, comparing banks, **mismatched periods**, loan balance & interest/principal
- **Ch. 4 (next):** average inflation rate, actual vs. constant dollars, real (inflation-free) rate

Always: draw the diagram $\rightarrow$ align the rate to the payment period $\rightarrow$ show factor notation before numbers $\rightarrow$ calculate the numbers.

Before Class

To do before our class meeting:

1. **Read Sections 3.1–3.3** (nominal/effective rates; mismatched periods)
2. Complete the **Lecture 8 pre-class quiz** on Blackboard before class

In class: We will do some practice problems in class today.