ISE 307: Engineering Economic Analysis

Lecture 7: Gradient Series and Composite Cash Flows (Chapter 2)

Term 253 (Summer)

Instructor: Mansur M. Arief

Industrial and Systems Engineering, KFUPM

Office: 22-219 — Email: mansur.arief@kfupm.edu.sa

Lecture 7

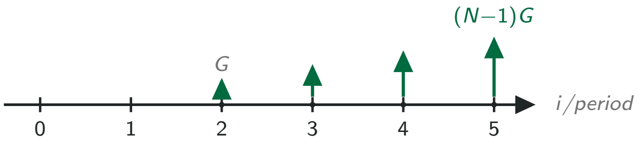
Gradients and the Decomposition Strategy for Composite Flows

Lecture 7 Objectives

1. Recognize a **strict linear gradient** and value it with $(P/G, i, N)$
2. Decompose a realistic growing series into **uniform** + **gradient** blocks
3. Value a **geometric gradient** (growth or decline at rate g)
4. Attack any **composite** cash flow by decomposition

The Strict Linear Gradient $(P/G, i, N)$

The first cash flow is ZERO



$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right] = G\,(P/G,\, i,\, N)$$

- The strict gradient is $0,\ G,\ 2G,\ \ldots,\ (N{-}1)G$ — **nothing at** $n = 1$
- (P/G) places the value of the whole staircase at $n = 0$

Example: A Rising Maintenance Cost

Growth of $G = 1{,}000$ SAR/yr for 5 years, $i = 10\%$

A pump's maintenance cost is 0 in year 1 and rises by 1,000 SAR each year, reaching 4,000 in year 5. Find the present worth.

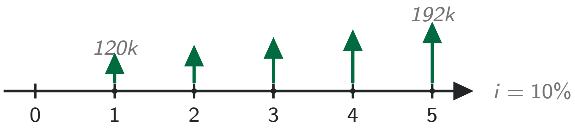
- Set up: $P = G\,(P/G,\,10\%,\,5) = 1{,}000\,(P/G,\,10\%,\,5)$
- Read the row $N = 5$ in the 10% table: $(P/G,\,10\%,\,5) = 6.8618$
- Multiply: $P = 1{,}000 \times 6.8618 = \boxed{6{,}862 \text{ SAR}}$

A pure gradient $0, 1{,}000, 2{,}000, 3{,}000, 4{,}000$ is worth 6,862 SAR today — but real costs rarely start at zero.

Realistic Growth = Uniform + Gradient

Decompose, then add

A project's revenue is 120k SAR in year 1, growing +**18k SAR/yr** till year 5 ($i = 10\%$):



Split into a flat 120,000 series + a strict 18,000 gradient:

$$\begin{aligned} P &= 120{,}000\,(P/A,\,10\%,\,5) + 18{,}000\,(P/G,\,10\%,\,5) \\ &= 120{,}000\,(3.7908) + 18{,}000\,(6.8618) \\ &= 454{,}896 + 123{,}512 = \boxed{\text{SAR } 578{,}408} \end{aligned}$$

Interest Factor Table at $i = 10\%$

N	(P/F)	(P/A)	(P/G)
1	0.9091	0.9091	0.0000
2	0.8264	1.7355	0.8264
3	0.7513	2.4869	2.3291
4	0.6830	3.1699	4.3781
5	**0.6209**	**3.7908**	**6.8618**
6	0.5645	4.3553	9.6842
8	0.4665	5.3349	16.0287
10	0.3855	6.1446	22.8913

How to use it:

- (P/A) values the **flat base**
- (P/G) values the **growth on top**
- Both at the same i and N, then add

Note: (P/G) *at* $N = 1$ *is always 0 — the gradient has no flow in year 1.*

Geometric Gradient

When growth is proportional, i.e., not additive.

$$A_n = A_1(1+g)^{n-1}$$

$$P = \begin{cases} A_1\left[\dfrac{1-(1+g)^N(1+i)^{-N}}{i-g}\right] & \text{if } i \neq g \\ A_1\dfrac{N}{1+i} & \text{if } i = g \end{cases}$$

A revenue stream is **120,000 SAR in year 1, growing 15%/yr** for 5 years, $i = 10\%$:

$$P = 120{,}000\left[\frac{1-(1.15)^5(1.10)^{-5}}{0.10-0.15}\right] = 120{,}000\,(4.9779) = \boxed{597{,}347\ \mathrm{SAR}}$$

Geometric Gradient for Decline

Case when $g < 0$

- The same formula holds.
- For example: An op-cost of 25,000 falling 5%/yr for 5 yrs at $i = 10\%$ has:

$$P = 25{,}000 \left[\frac{1 - (1.05)^5 (1.10)^{-5}}{0.10 - (-0.05)}\right] = 25{,}000\,(3.4636) = \boxed{\text{SAR } 86{,}590}$$

Composite Cash Flows: The Strategy

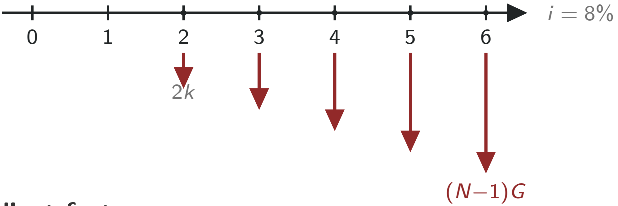
Any messy stream is a sum of four block types you now own:

Block	**Tool**	**Watch out for**
Single flow	$(P/F),(F/P)$	count the periods carefully
Uniform series	$(P/A),(F/A)$	(P/A) lands 1 period early
Linear gradient	(P/G)	first flow is 0
Geometric	A_1 formula	the $i=g$ special case

Workflow: **draw** the diagram $\rightarrow$ **slice** into blocks $\rightarrow$ value each at a **common date** $\rightarrow$ **add**.

Practice Problem 1

A conveyor's repair cost is 0 in year 1 and rises by 2,000 SAR/yr through year 6 at $i = 8\%$. Find the present worth of the repairs.



Apply the gradient factor:

$$P = 2{,}000\,(P/G,\,8\%,\,6) = 2{,}000\,(10.5233)$$
$$= \boxed{\text{SAR } 21{,}047}$$

Practice Problem 2

Output revenue is 80,000 SAR in year 1, growing +5,000 SAR/yr through year 6 at $i = 8\%$. Find the present worth.

Split into a flat 80,000 series + a strict 5,000 gradient:

$$\begin{aligned} P &= 80{,}000\,(P/A,\,8\%,\,6) + 5{,}000\,(P/G,\,8\%,\,6) \\ &= 80{,}000\,(4.6229) + 5{,}000\,(10.5233) \\ &= 369{,}832 + 52{,}617 = \boxed{\text{SAR } 422{,}449} \end{aligned}$$

The base 80,000 carries most of the value.

Practice Problem 3

An energy bill is 40,000 SAR in year 1 and rises **8%/yr** for 5 years at $i = 12\%$. Find the present worth.

Use the geometric-gradient formula ($i \neq g$):

$$P = 40{,}000\left[\frac{1 - (1.08)^5(1.12)^{-5}}{0.12 - 0.08}\right]$$
$$= 40{,}000\,(4.1566) = \boxed{\text{SAR } 166{,}264}$$

Check: if growth were 0, $P = 40{,}000\,(P/A, 12\%, 5) = 40{,}000(3.6048) = 144{,}192$. *The 8% growth raises it, as expected.*

Summary

- Strict linear gradient $0, G, \ldots, (N{-}1)G$: value with $(P/G, i, N)$; **ZERO first flow**
- Realistic growth $=$ uniform series $+$ strict gradient (decompose, then add)
- Geometric gradient: $P = A_1\left[1 - (1+g)^N (1+i)^{-N}\right]/(i-g)$; handles decline with $g < 0$
- Composite flows: draw $\rightarrow$ slice $\rightarrow$ value at a common date $\rightarrow$ add

Before Class

To do before our class meeting:

1. **Read Sections 2.4–2.6** (gradients and composite flows)
2. Complete the **Lecture 7 pre-class quiz** on Blackboard before class
3. Try this: *revenue 60,000 in year 1, +4,000/yr for 5 yrs at 9% — find P*

In class: we'll practice a composite cash flow for each team's project.